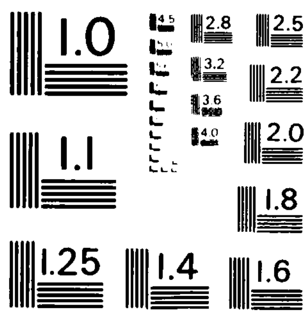
AD-A131 523 LIKELIHOOD RATIO TESTS ON COVARIANCE MATRICES AND MEAN VECTORS OF COMPLEX..(U) PITTSBURGH UNIV PA CENTER FOR MULTIVARIATE ANALYSIS P R KRISHNAIAN ET AL. MAR 83 1/1

UNCLASSIFIED AFOSR-TR-83-0692 F49629-82-K-0001 F/G 12/1 NL

END
DATE
FILMED
9 83
DTIC



MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS - 1963 - A

AFOSR-TR- 83-0692

AD A 131523

Center for Multivariate Analysis

University of Pittsburgh

DTIC FILE COPY

Approved for public release; distribution unlimited.

DTIC ELECTE 1983 D

E

83 08 19 056

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER AFOSR-TR- 83-0692	2. GOVT ACCESSION NO. A131523
3. RECIPIENT'S CATALOG NUMBER	
4. TITLE (and Subtitle) "LIKELIHOOD RATIO TESTS ON COVARIANCE MATRICES AND MEAN VECTORS OF COMPLEX MULTIVARIATE NORMAL POPULATIONS AND THEIR APPLICATIONS IN TIME SERIES"	5. TYPE OF REPORT & PERIOD COVERED Technical
	6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) J.C. Lee, T.C. Chang	8. CONTRACT OR GRANT NUMBER(s) F49629-82-K-0001
9. PERFORMING ORGANIZATION NAME AND ADDRESS University of Pittsburgh, Pittsburgh, PA 15260	10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS PE61102F; 2304/A5
11. CONTROLLING OFFICE NAME AND ADDRESS AFOSR/NM, Bldg. 410, Bolling AFB DC 20332	12. REPORT DATE March 1983
	13. NUMBER OF PAGES 67
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)	15. SECURITY CLASS. (of this report) UNCLASSIFIED
	15a. DECLASSIFICATION DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Complex multivaritate normal, homogeneity of populations, likelihood ratio tests, multiple independence, multiple time series in the frequency domain, and tables.

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

In this paper, the authors reviewed the literature on computational aspects of the distributions of the likelihood ratio statistics for testing various hypotheses on the covariance matrices and mean vectors of complex multivariate normal populations. Applications of some of these test procedures in the area of inference on multiple time series in the frequency domain are also discussed. In the Appendix, the authors give tables which are useful in implementation of various likelihood ratio test statistics discussed in this paper.

DD FORM 1 JAN 73 1473

LIKELIHOOD RATIO TESTS ON COVARIANCE MATRICES AND MEAN VECTORS OF COMPLEX MULTIVARIATE NORMAL POPULATIONS AND THEIR APPLICATIONS IN TIME SERIES

P. R. Krishnaiah*

University of Pittsburgh

J. C. Lee

Bell Telephone Laboratories

T. C. Chang

University of Cincinnati

March 1983

Technical Report No. 83-03

Center for Multivariate Analysis
University of Pittsburgh
Ninth Floor, Schenley Hall
Pittsburgh, PA 15260

*The work of this author is sponsored by the Air Force Office of Scientific Research under Contract F49629-82-K0001. Reproduction in whole or in part is permitted for any purpose of the United States Government.

ABSTRACT

In this paper, the authors reviewed the literature on computational aspects of the distributions of the likelihood ratio statistics for testing various hypotheses on the covariance matrices and mean vectors of complex multivariate normal populations. Applications of some of these test procedures in the area of inference on multiple time series in the frequency domain are also discussed. In the Appendix, the authors give tables which are useful in implementation of various likelihood ratio test statistics discussed in this paper.

Key words and phrases:

Complex multivariate normal, homogeneity of populations, likelihood ratio tests, multiple independence, multiple time series in the frequency domain, and tables.

1. Introduction

It is known that certain estimates of spectral density matrices of the stationary and Gaussian multiple time series are distributed approximately as complex Wishart matrices. So, complex multivariate distributions are useful (e.g., see Brillinger (1974), and Hannan (1970)) in the area of inference on multiple time series. These distributions are useful in the area of nuclear physics (see Carmeli (1974)) also.

Wooding (1956) introduced the complex multivariate normal distribution. A complex random vector is said to be distributed as a complex multivariate normal if its real and imaginary parts are distributed jointly as a multivariate normal with a structered covariance matrix. Motivated by applications in time series, Goodman (1963a, b) made a systematic study of the complex multivariate normal distribution and complex Wishart matrix. Since then, James (1964), Khatri (1965), Krishnaiah (1976) and other workers in the field have investigated various aspects of complex multivariate distributions. For a review of the literature on complex multivariate distributions, the reader is refered to Krishnaiah (1976). In this paper, we review the literature on the likelihood ratio tests on mean vectors and covariance matrices of the complex multivariate normal populations as well as some of their applications in the area of inference on multiple time series in the frequency domain.

In Section 2 of this paper, we discuss the complex multivariate normal and complex Wishart matrix. The distribution of the determinant of the complex multivariate beta matrix is discussed in Section 3, whereas Section 4 is devoted to the likelihood ratio test procedure for testing the hypothesis of multiple independence of several sets of variables when their joint distribution is complex multivariate normal. Likelihood ratio tests for the hypothesis of

sphericity and the hypothesis specifying the covariance matrix are discussed in Sections 5 and 6 respectively. In Section 7 we discuss the likelihood ratio test for the homogeneity of the covariance matrices whereas the likelihood ratio test procedure for the homogeneity of several complex multivariate normal populations is discussed in Section 8. Likelihood ratio test procedure specifying the covariance matrix and mean vector is discussed in Section 9. Applications of some test procedures on the covariance matrices of the complex multivariate normal populations to the area of inference on multiple time series in the frequency domain are discussed in Section 10. Various tables useful in implementation of certain likelihood ratio test procedures are given in the Appendix. These tables are constructed by approximating a suitable power of the likelihood ratio statistics with Pearson's Type I distribution by using the first four moments. The accuracy of these tables is found to be good.

2. Complex Multivariate Normal and Complex Wishart Distributions

Let $\underset{\sim}{z} = \underset{\sim}{x} + i\underset{\sim}{y}$ where $\underset{\sim}{x}$ and $\underset{\sim}{y}$ are of order $p\times 1$ and $(\underset{\sim}{x}', \underset{\sim}{y}')$ is distributed as 2p-variate normal with mean vector $(\underset{\sim}{\mu}_1', \underset{\sim}{\mu}_2')$ and covariance matrix

$$C = \begin{pmatrix} \Sigma_1 & -\Sigma_2 \\ \Sigma_2 & \Sigma_1 \end{pmatrix} \tag{2.1}$$

where A' denotes the transpose of A. Then, the distribution of $\underset{\sim}{z}$ is known to be the complex multivariate normal distribution with mean vector $\underset{\sim}{\mu}$ and covariance matrix Σ where $\underset{\sim}{\mu} = \underset{\sim}{\mu}_1 + i\underset{\sim}{\mu}_2$ and $\Sigma = 2(\Sigma_1 - i\Sigma_2)$. The probability density function (p.d.f.) of $\underset{\sim}{z}$ is given by

$$f(\underset{\sim}{z}) = \frac{1}{\pi^p|\Sigma|} \exp\{-(\overline{\underset{\sim}{z}-\underset{\sim}{\mu}})'\Sigma^{-1}(\underset{\sim}{z}-\underset{\sim}{\mu})\} \tag{2.2}$$

whereas the characteristic function of $\underset{\sim}{z}$ is

$$\phi(t) = \exp\{i\bar{\underset{\sim}{t}}'\underset{\sim}{\mu} - \frac{1}{4}\bar{\underset{\sim}{t}}'\Sigma^{-1}\underset{\sim}{t}\} \tag{2.3}$$

where $\underset{\sim}{t} = \underset{\sim}{t}_1 + i\underset{\sim}{t}_2$ and $\bar{\underset{\sim}{t}}$ is the complex conjugate of $\underset{\sim}{t}$. Wooding (1956) derived expressions for the p.d.f. and characteristic function of $\underset{\sim}{z}$. The

Maximum likelihood estimates of $\underset{\sim}{\mu}$ and Σ based on a random sample $(z_1, \ldots \underset{\sim}{z}_n)$ are known to be

$$\begin{aligned} \hat{\underset{\sim}{\mu}} &= N^{-1}\sum_{j=1}^{N} \underset{\sim}{z}_j \\ \hat{\Sigma} &= N^{-1}\sum_{j=1}^{N} (\underset{\sim}{z}_j - \hat{\underset{\sim}{\mu}})(\underset{\sim}{z}_j - \hat{\underset{\sim}{\mu}})' \end{aligned} \tag{2.4}$$

Also, $\hat{\underset{\sim}{\mu}}$ and $\hat{\Sigma}$ are distributed independent of each other.

Next, let $S = N\hat{\Sigma}$. Then, the distribution of S is known to be a central complex Wishart matrix with $n=N-1$ degrees of freedom. The probability density of S is known (Goodman (1963b)) to be

$$f(S) = \frac{|S|^{n-p}\, \operatorname{etr}\{-\Sigma^{-1}S\}}{\pi^{p(p-1)/2} \prod_{j=1}^{p} \Gamma(n-j+1)\, |\Sigma|^{n}} \tag{2.5}$$

where etr B denotes the exponential of the trace of the matrix B.

3. Distribution of the Determinant of the Complex Multivariate Beta Matrix

In this section, we discuss the distribution of the determinant of the complex multivariate beta matrix. This distribution is useful for testing the hypothesis of the equality of several mean vectors and the equality of two covariance matrices when the underlying distributions are complex multivariate normal. It is also useful in testing the hypothesis $H_1:\Sigma_{12}=0$ where Σ_{12} is the covariance between two sets of variables whose joint distribution is complex multivariate normal.

Let A_1: $p\times p$ and A_2: $p\times p$ be independently distributed as the central complex Wishart matrices with n and q degrees of freedom, and let $E(A_1/n) = E(A_2/q) = \Sigma$. Then $A_1(A_1+A_2)^{-1}$ is known to be a (central) complex multivariate beta matrix. Now, let

$$U = \left|A_1(A_1+A_2)^{-1}\right|. \tag{3.1}$$

The h-th moment of U is known to be

$$E(U^h) = \prod_{j=1}^{p}\left|\frac{\Gamma(n+h-j+1)\;\Gamma(n+q-j+1)}{\Gamma(n-j+1)\;\Gamma(n+h+q-j+1)}\right|. \tag{3.2}$$

Using the first four moments of $U^{1/b}$, Lee, Krishnaiah and Chang (1975) have approximated the distribution of $U^{1/b}$ with the Pearson's Type I distribution where b is a properly chosen integer. The constant b is chosen to be equal to 1 or 2 according as $M>20$ or $M<20$ where $M = n-p+1$. By making use of this approximation, values of c_1 are computed, where

$$P[C_1\le c_1] = (1-\alpha), \tag{3.3}$$

$C_1 = -(2n+q-p)\log U/\chi^2_{2pq,\alpha}$, and $\chi^2_{2pq,\alpha}$ is the upper 100 α°/o value of

χ^2 with 2pq degrees of freedom. The values of c_1 are computed for $\alpha = 0.005$, 0.01. 0.025, 0.05, 0.1, 0.90, 0.95, 0.99, 0.995, M = 1(1)10(2)20, 30, 60, 120, where M = n-p+1. These values are given in a technical report by Lee, Krishnaiah and Chang (1975). The upper 5% and 1% points are reproduced in Table 7 of this chapter. To check for the accruacy of the entries in Table 7, the above authors compared some of the values obtained by the Pearson type approximation with the corresponding exact values. These comparisons are given in Table 1.

Table 1

Comparison of the Pearson type Approximation with exact expression for the Distribution of c_1

	p = 2 q = 3			p = 2 q = 20		
M	α	L-K-C	Exact	α	L-K-C	Exact
1	0.05	1.286	1.289	0.05	1.928	1.932
1	0.01	1.350	1.349	0.01	2.085	2.080
5	0.05	1.029	1.029	0.05	1.243	1.243
5	0.01	1.033	1.033	0.01	1.262	1.262
9	0.05	1.010	1.011	0.05	1.128	1.128
9	0.01	1.012	1.012	0.01	1.137	1.137

The constant α in Table 1 is defined by Eq. (3.3). Also, the values under the column "L-K-C" are the values of c_1 obtained by Lee, Krishnaiah and Chang (1975) using the Pearson type approximation whereas the values under the column "Exact" are the corresponding values given by Gupta (1971). Table 1 indicates that the accruacy of the Pearson type approximation is sufficient for practical purposes.

4. Test for Independence of Sets of Variates

Let $\underset{\sim}{z}' = (\underset{\sim}{z}_1', \ldots, \underset{\sim}{z}_q')$ be distributed as a complex multivariate normal distribution with mean vector $\underset{\sim}{\mu}' = (\underset{\sim}{\mu}_1', \ldots, \underset{\sim}{\mu}_q')$ and covariance matrix Σ. Also, let $E\left\{(\underset{\sim}{z}_i - \underset{\sim}{\mu}_i)\,\overline{(\underset{\sim}{z}_j - \underset{\sim}{\mu}_j)}'\right\} = \Sigma_{ij}$, and $E(\underset{\sim}{z}_i) = \underset{\sim}{\mu}_i$. It is assumed that $\underset{\sim}{z}_i$ is of order $p_i \times 1$ and $p_1 + \ldots + p_q = s$. In this section we discuss the problem of testing the hypothesis H_2 where

$$H_2 : \Sigma_{ij} = 0 \tag{4.1}$$

for $i \neq j = 1, \ldots, q$. Now let

$$A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1q} \\ A_{21} & A_{22} & \cdots & A_{2q} \\ \vdots & \vdots & \vdots & \vdots \\ A_{q1} & A_{q2} & \cdots & A_{qq} \end{bmatrix}$$

where

$$A_{gh} = \sum_{j=1}^{N} (\underset{\sim}{z}_{gj} - \underset{\sim}{z}_{g.})\overline{(\underset{\sim}{z}_{hj} - \underset{\sim}{z}_{h.})}', \quad \underset{\sim}{z}_{g.} = N^{-1} \sum_{j=1}^{N} \underset{\sim}{z}_{gj}$$

and $(\underset{\sim}{z}_{1j}', \ldots, \underset{\sim}{z}_{qj}')$ is j-th independent observation on $(\underset{\sim}{z}_1', \ldots, \underset{\sim}{z}_q')$. The likelihood ratio statistic for testing H_2 is

$$\lambda_2 = \frac{|A|}{\prod_{j=1}^{q} |A_{jj}|} . \tag{4.2}$$

In the analogous real case, Wilks (1935) derived the likelihood ratio test for multiple independence. The hth moment of λ_2 is

$$E(\lambda_2^h) = \left[\prod_{j=1}^{s} \frac{\Gamma(n+h-j+1)}{\Gamma(n-j+1)}\right]\left[\prod_{i=1}^{q} \prod_{\alpha=1}^{p_i} \frac{\Gamma(n-\alpha+1)}{\Gamma(n+h-\alpha+1)}\right] \tag{4.3}$$

where $n = N-1$ and $\Gamma(\cdot)$ is the complete gamma function. The distribution of $\lambda_2^{1/4}$ is approximated by Pearson's type I distribution with density

$$g(x) = \{\beta(\alpha+1,\varepsilon+1)(\sigma_1-\sigma_0)^{\alpha+\varepsilon+1}\}^{-1}(x-\sigma_0)^{\alpha}(\sigma_1-x)^{\varepsilon} \tag{4.4}$$

where $\sigma_0 < x < \sigma_1$ and α and ε are some real numbers.

Approximate percentage points of the distribution of $\tilde{\lambda}_2 = -2 \log \lambda_2$ are constructed by Krishnaiah, Lee and Chang (1975, 1976) for $p_i=p=1,2,3$; $q=3,4,5$; $\alpha = 0.01, 0.05, 0.10$; $M = 1(1)20(2)30$ where $M = n-s-3$, and $\Pr[\tilde{\lambda}_2 \leq c_2 | H_2] = (1-\alpha)$. These percentage points are reproduced in Table 8. Percentage points for $q=2$ are given in Table 7.

Now, consider a class of statistics $W(0 \leq W \leq 1)$ whose moments are of the form

$$E\{W^h\} = K \prod_{j=1}^{c} y_j^{y_j} \prod_{k=1}^{a} x_k^{x_k} \frac{\prod_{k=1}^{a} \Gamma[x_k(1+h)+\xi_k]}{\prod_{j=1}^{c} \Gamma[y_j(1+h)+y_j]}, \quad h = 0,1,\ldots \tag{4.5}$$

where K is a normalizing constant such that $E\{W^o\} = 1$ and $\sum_{k=1}^{a} x_k = \sum_{j=1}^{c} y_j$.

The likelihood ratio test statistics considered in this chapter are special cases of the above class of statistics. Box (1949) gave explicitly the first few terms of an asymptotic expression for the distribution of a class of statistics whose moments are of the form (4.5). But the first few terms alone are not sufficient to get the desired degree of accuracy in a number of practical situations. So, Lee, Krishnaiah and Chang (1976) gave terms up to order n^{-15} explicitly. In Table 2, given below, a comparison of the values obtained by using Pearson type approximation is made with the corresponding values obtained by the asymptotic expression of order n^{-13}.

Table 2

Comparison of the Pearson Type Approximation with the Asymptotic Expansion for the Distribution of $\tilde{\lambda}_2$

n	q = 3	p = 1		q = 4	p = 2		q = 5	p = 1	
	c_2	α_1	α_2	c_2	α_1	α_2	c_2	α_1	α_2
10	1.459	0.05	0.0499	-	-	-	4.011	0.05	0.0487
10	1.949	0.01	0.0100	-	-	-	4.811	0.01	0.0095
15	0.923	0.05	0.0500	5.733	0.05	0.0479	2.435	0.05	0.0497
15	1.233	0.01	0.0100	6.496	0.01	0.0093	2.914	0.01	0.0099
20	0.675	0.05	0.0500	3.958	0.05	0.0493	1.752	0.05	0.0499
20	0.902	0.01	0.0100	4.479	0.01	0.0098	2.096	0.01	0.0100
30	0.439	0.05	0.0501	2.455	0.05	0.0498	1.124	0.05	0.0500
30	0.587	0.01	0.0100	2.777	0.01	0.0099	1.344	0.01	0.0100

In this table, α_1 is the value of α if we use the Pearson type approximation and α_2 is the value of α if we use the asymptotic expression of order n^{-13}. From the table, we observe that the accuracy of the Pearson type approximation is sufficient for practical purposes.

5. Test for Sphericity

The likelihood ratio statistic for testing H_3: $\Sigma = \sigma^2 \Sigma_0$ is given by

$$\lambda_3 = \frac{|A\,\Sigma_o^{-1}|}{(\mathrm{tr}\,A\,\Sigma_o^{-1}/s)^s} \tag{5.1}$$

where Σ_o is known, A was defined in Section 4 and trA denotes the trace of A. The h^{th} moment of λ_3 is known to be

$$E(\lambda_3^h) = \frac{s^{hs}\,\Gamma\,(sn)}{\Gamma(sn + sh)} \prod_{j=1}^{s} \frac{\Gamma(n+h-j+1)}{\Gamma(n-j+1)} \quad . \tag{5.2}$$

Mauchly (1940) derived the likelihood ratio statistic for testing the hypothesis of sphericity when the underlying distribution is real multivariate normal.

The distribution of $\lambda_3^{1/b}$ is approximated by a Pearson Type I distribution, where b is a suitably chosen integer. For $M \geq 22$, we took $b=2$ and for $M<22$, $b=4$. Using the approximation described above, approximate upper percentage points of distribution of $\tilde{\lambda}_3 = -2 \log \lambda_3$ were constructed by Krishnaiah, Lee and Chang (1975, 1976) for $s = 2(1)10$, $\alpha = 0.01, 0.05$, $M = 1(1)\,20(2)30(5)50,60$, where $M = n-s-3$ and $P[\tilde{\lambda}_3 \leq c_3 | H_3] = (1 - \alpha)$. These values are reproduced in Table 9.

In Table 3, we compare the values obtained by the Pearson type approximation with the corresponding values obtained by using Box's asymptotic expression of order n^{-13}. Upper percentage points of the distribution of $\tilde{\lambda}_3$ for $\alpha = 0.01, 0.05$ and $s = 3(1)6$ are also given by Nagarsenker and Das (1975).

Table 3

Comparison of the Pearson Type Approximation with the Asymptotic Expression for the Distribution of $\tilde{\lambda}_3$

	s = 5			s = 8		
n	c_3	α_1	α_2	c_3	α_1	α_2
15	2.763	0.05	0.0496	-	-	-
15	3.265	0.01	0.0099	-	-	-
21	1.895	0.05	0.0498	4.565	0.05	0.0490
21	2.238	0.01	0.0100	5.093	0.01	0.0097
41	0.928	0.05	0.0500	2.160	0.05	0.0499
41	1.095	0.01	0.0100	2.409	0.01	0.0100
51	0.739	0.05	0.0500	1.711	0.05	0.0499
51	0.873	0.01	0.0100	1.908	0.01	0.0100

In the above table, α_1 is the value of α obtained by using the Pearson type approximation whereas α_2 is the value of α obtained by using Box's asymptotic expansion of order n^{-13}. This table indicates that the accuracy of Pearson type approximation is sufficient for practical purposes.

6. Test Specifying the Covariance Matrix

The modified likelihood ratio statistic for testing the hypothesis H_4: $\Sigma = \Sigma_o$ is given by

$$\lambda_4 = (e/n)^{sn} \, |A\Sigma_o^{-1}|^n \, \mathrm{etr}(-A\Sigma_o^{-1}). \tag{6.1}$$

The modified likelihood ratio test statistic is obtained from the likelihood ratio test statistic by changing N to n. The moments of λ_4 are seen to be

$$E(\lambda_4^h) = (e/n)^{shn} \, |\Sigma_o|^{nh} \, |I + h\Sigma_o|^{-n(1+h)}$$

$$\times \prod_{i=1}^{s} \{\Gamma(n+nh+1-i)/\Gamma(n+1-i)\}. \tag{6.2}$$

Anderson (1958) derived the likelihood ratio statistic for testing the hypothesis that the covariance matrix is equal to a specified matrix when the underlying distribution is real multivariate normal. The distribution of $\lambda_4^{1/b}$ can be approximated with a Pearson Type I distribution using the first four moments, where b is a suitably chosen integer. Using the above approximation, Krishnaiah, Lee and Chang (1975, 1976) computed the percentage points of the distribution of $\tilde{\lambda}_4 = -2 \log \lambda_4$ for $s = 2(1)10$, $\alpha = 0.01, 0.05$, $M=1(1)20(2)30$, where $M = n-s-1$ and $P[\tilde{\lambda}_4 \le c_4 | H_4] = (1-\alpha)$. These percentage points are reproduced in Table 10.

7. Test for Multiple Homogeneity of the Covariance Matrices

Let $\underset{\sim}{z}_1,\ldots,\underset{\sim}{z}_q$ be independently distributed as complex p-variate normal with mean vectors $\underset{\sim}{\mu}_1,\ldots,\underset{\sim}{\mu}_q$ and covariance matrices $\Sigma_{11},\ldots,\Sigma_{qq}$, respectively. Also, let $\underset{\sim}{z}_{ij}(j=1,\ldots,N_i)$ be the j^{th} independent observation on $\underset{\sim}{z}_i$. In this section, we study the Pearson type approximation to the distribution of the likelihood ratio statistic for testing H_5 where

$$H_5: \begin{cases} \Sigma_{11}=\ldots=\Sigma_{q_1q_1} \\ \Sigma_{q_1+1,q_1+1}=\ldots=\Sigma_{q_2^*,q_2^*} \\ \vdots \qquad\qquad \vdots \quad \vdots \\ \Sigma_{q_{d-1}^*+1,q_{d-1}^*+1}=\ldots=\Sigma_{qq} \end{cases}$$

$q_0^* = 0$, $q_1^* = q_1$, $q_d^* = q$ and $q_j^* = q_1+\ldots+q_j$. The modified likelihood ratio statistic (obtained by changing N_i to n_i in the likelihood ratio test statistic) for testing H_5, is given by

$$\lambda_5 = \frac{\prod_{i=1}^{q} |A_{ii}/n_i|^{n_i}}{\prod_{j=1}^{d} \left|\sum_{i=q_{j-1}^*+1}^{q_j^*} A_{jj}/n_j^*\right|^{n_j^*}} \tag{7.1}$$

where $n_i = N_i-1, n_j^* = \sum_{i=q_{j-1}^*+1}^{q_j^*} n_i$ and

$$A_{ii} = \sum_{j=1}^{N_i} (\underset{\sim}{z}_{ij} - \underset{\sim}{z}_{i.})(\overline{\underset{\sim}{z}_{ij} - \underset{\sim}{z}_{i.}})', \quad \underset{\sim}{z}_{i.} = \sum_{j=1}^{N_i} \underset{\sim}{z}_{ij}/N_i .$$

The moments of λ_5 are given by

$$E(\lambda_5^h) = \left[\frac{\prod_{\alpha=1}^{d} n_\alpha^{* \, phn_\alpha^*}}{\prod_{\alpha=1}^{q} (n_\alpha)^{phn_\alpha}}\right] \prod_{i=1}^{p} \prod_{\alpha=1}^{d} \left| \prod_{g=q_{\alpha-1}^*+1}^{q_\alpha^*} \frac{\Gamma(n_g+hn_g+1-i)}{\Gamma(n_g+1-i)} \right|$$

$$\times \frac{\Gamma(n_\alpha^*+1-i)}{\Gamma(n_\alpha^*+hn_\alpha^*+1-i)} . \tag{7.2}$$

Using the first four moments of λ_5, the distribution of $\lambda_5^{1/b}$ can be approximated with a Pearson's Type I distribution where b is a suitably chosen integer. This approximation was used by Krishnaiah, Lee and Chang (1975, 1976) to compute approximate percentage points of the distribution of $\tilde{\lambda}_5 = -2 \log \lambda_5$ for $n_i = n_o$, $q = dk$ (i.e., there are k populations in each of the d groups). These points are reproduced in Table 11 for d=1.

In Tables 4 and 5, we compare the values obtained by the Pearson Type approximation for the distribution function of $\tilde{\lambda}_5$ with the corresponding values obtained by using the Box's asymptotic expansion up to terms of order n^{-13}. In these tables, the constant c_5 is defined as

$$P[-2 \log \lambda_5 \leq c_5 \,|H_5] = (1-\alpha). \tag{7.3}$$

Also, α_1 is the value of α if we use the Pearson type approximation whereas α_2 is the value of α if we use the asymptotic expression of order n^{-13}. Tables 4 and 5 indicate that the accuracy of the Pearson type approximation is sufficient for practical purposes.

In the real case, Wilks (1932) derived the likelihood ratio statistic for testing the homogeneity of the covariance matrices whereas Krishnaiah and Lee (1976) discussed how certain tests of hypotheses on linear structure of the covariance matrices can be reduced to the problem of testing for the multiple homogeneity of the covariance matrices.

Table 4

Comparison of the Pearson Type Approximation with the Asymptotic Expansion for the Distribution of $\tilde{\lambda}_5$ when $d = 1$

		$p = 3$			$p = 4$		
n_0	q	c_5	α_1	α_2	c_5	α_1	α_2
10	2	19.82	0.05	0.0501	33.00	0.05	0.0502
10	2	25.40	0.01	0.0100	40.21	0.01	0.0101
10	6	69.68	0.05	0.0500	12.14	0.05	0.0492
10	6	79.11	0.01	0.0100	13.39	0.01	0.0098
15	2	18.72	0.05	0.0501	30.33	0.05	0.0500
15	2	23.99	0.01	0.0100	36.93	0.01	0.0100
15	6	66.70	0.05	0.0500	113.77	0.05	0.0498
15	6	75.69	0.01	0.0100	125.48	0.01	0.0099
20	2	18.23	0.05	0.0500	29.18	0.05	0.0500
20	2	23.35	0.01	0.0100	35.52	0.01	0.0100
20	6	65.34	0.05	0.0499	110.45	0.05	0.0499
20	6	74.13	0.01	0.0100	121.78	0.01	0.0100

Table 5

Comparison of the Pearson Type Approximation with Asymptotic Expansion for the Distribution of $\tilde{\lambda}_5$ when $d > 1$

			p = 1			p = 2			p = 3			p = 4		
n_0	q	d	c_5	α_1	α_2	c_5	α_1	α_2	c_5	α_1	α_2	c_5	α_1	α_2
10	6	3	8.01	0.05	0.0500	23.08	0.05	0.0500	46.97	0.05	0.0501	81.73	0.05	0.0503
10	6	3	11.62	0.01	0.0100	28.78	0.01	0.0100	55.03	0.01	0.0100	92.47	0.01	0.0101
10	6	2	9.70	0.05	0.0500	28.56	0.05	0.0500	58.66	0.05	0.0499	102.21	0.05	0.0500
10	6	2	13.57	0.01	0.0100	34.76	0.01	0.0100	67.44	0.01	0.0100	113.93	0.01	0.0100
20	6	3	7.91	0.05	0.0500	22.00	0.05	0.0500	43.22	0.05	0.0500	72.32	0.05	0.0500
20	6	3	11.49	0.01	0.0100	27.43	0.01	0.0100	50.60	0.01	0.0100	81.77	0.01	0.0100
20	6	2	9.59	0.05	0.0500	27.37	0.05	0.0500	54.49	0.05	0.0500	91.76	0.05	0.0500
20	6	2	13.42	0.01	0.0100	33.31	0.01	0.0100	62.63	0.01	0.0100	102.25	0.01	0.0100
30	6	3	7.88	0.05	0.0500	21.66	0.05	0.0500	42.12	0.05	0.0500	69.73	0.05	0.0500
30	6	3	11.44	0.01	0.0100	27.01	0.01	0.0100	49.31	0.01	0.0100	78.85	0.01	0.0100
30	6	2	9.56	0.05	0.0500	27.00	0.05	0.0500	53.26	0.05	0.0500	88.85	0.05	0.0500
30	6	2	13.38	0.01	0.0100	32.86	0.01	0.0100	61.22	0.01	0.0100	98.98	0.01	0.0100

8. Simultaneous Tests for the Homogeneity of Populations

In this section, we discuss the likelihood ratio test for the homogeneity of complex multivariate normal populations. The hypothesis of the homogeneity of the q complex multivariate distributions defined in Section 7 is equivalent to the hypothesis H_6 where

$$H_6: \begin{cases} \Sigma_{11} = \dots = \Sigma_{qq} \\ \underset{\sim}{\mu}_1 = \dots = \underset{\sim}{\mu}_q . \end{cases} \tag{8.1}$$

The modified likelihood ratio statistic for testing H_6 is given by

$$\lambda_6 = \frac{n^{pn} \prod_{i=1}^{q} |G_i|^{n_i}}{\prod_{i=1}^{q} n_i^{pn_i} \; |G + \sum_{i=1}^{q} N_i (\underset{\sim}{z}_{i.} - \underset{\sim}{z}_{..}) \overline{(\underset{\sim}{z}_{i.} - \underset{\sim}{z}_{..})}' |^{n}} \tag{8.2}$$

where $n = \sum_{i=1}^{q} n_i$, $n_i = N_i - 1$,

$$N = \sum_{i=1}^{q} N_i, \underset{\sim}{z}_{..} = \frac{1}{N} \sum_i \sum_j \underset{\sim}{z}_{ij}, \underset{\sim}{z}_{i.} = \frac{1}{N_i} \sum_{j=1}^{N_i} \underset{\sim}{z}_{ij}$$

$$G_i = \sum_{j=1}^{N_i} (\underset{\sim}{z}_{ij} - \underset{\sim}{z}_{i.}) \overline{(\underset{\sim}{z}_{ij} - \underset{\sim}{z}_{i.})}' , \qquad G = \sum_{i=1}^{q} G_i .$$

The moments of λ_6 are given by

$$E(\lambda_6^h) = [n^{phn} / \prod_{i=1}^{q} n_i^{phn_i}] \prod_{i=1}^{p} \prod_{j=1}^{q} \frac{\Gamma(n_j + hn_j + 1 - i)}{\Gamma(n_j + 1 - i)} \frac{\Gamma(n+q-i)}{\Gamma(n+hn+q-i)} \tag{8.3}$$

The distribution of $\lambda_6^{1/b}$ is approximated with a Pearson Type I distribution where b is a suitably chosen integer. Using this approximation, percentage points of the distribution of $\tilde{\lambda}_6 = -2 \log \lambda_6$ are computed by Chang, Krishnaiah and Lee (1975) for $\alpha = 0.01, 0.025, 0.05, 0.10$; $n_i = n_o$; $q = 2, 3, 4, 5$; $p = 1, 2, 3, 4$ and $M = n_o - p = 1(1)\ 20, 25, 30$. Table 12 gives the value of c_6 for $\alpha = 0.05, 0.01$ where c_6 is given by

$$P[\tilde{\lambda}_6 \le c_6 | H_6] = (1-\alpha). \quad (8.4)$$

To check for the accuracy of the entries in Table 12, we compared some of these values with the corresponding values obtained by using Box's asymptotic series of order n^{-13}. These comparisons are given in Table 6.

Table 6

Comparison of the Pearson Type Approximation with the Asymptotic Expansion

		p = 2			p = 3		
n_o	q	c_6	α_1	α_2	c_6	α_1	α_2
7	3	28.03	0.05	0.0499	50.49	0.05	0.0497
7	3	34.15	0.01	0.0100	58.80	0.01	0.0098
10	3	27.45	0.05	0.0500	48.06	0.05	0.0498
10	3	33.42	0.01	0.0100	55.92	0.01	0.0099
15	3	27.04	0.05	0.0500	46.46	0.05	0.0499
15	3	32.90	0.01	0.0100	54.04	0.01	0.0100
20	3	26.84	0.05	0.0500	45.73	0.05	0.0500
20	3	32.67	0.01	0.0100	53.18	0.01	0.0100

In the above table, α_1 is the value of α obtained by approximating $\lambda_6^{1/b}$ with Pearson's type I distribution whereas α_2 is the value of α detained by using Box's asymptotic series. Table 6 indicates that the accuracy of the values of α_1 is good.

9. Test Specifying the Values of the Covariance Matrix and Mean Vector

In this section, we consider the distribution of the likelihood ratio statistic for testing the hypothesis H_7, where

$$H_7: \begin{cases} \Sigma_{11} = \Sigma_o \\ \underset{\sim}{\mu}_1 = \underset{\sim}{\mu}_o \end{cases}$$

and Σ_o and $\underset{\sim}{\mu}_o$ are known. The likelihood ratio statistic for testing H_7 is given by

$$\lambda_7 = (e/N_1)^{pN_1} \; |G_1 \Sigma_o^{-1}|^{N_1} \; \mathrm{etr}[-\Sigma_o^{-1} \; \{G_1 + N_1(\underset{\sim}{z}_{1.} - \underset{\sim}{\mu}_o)(\overline{\underset{\sim}{z}_{1.} - \underset{\sim}{\mu}_o})'\}] \qquad (9.1)$$

The moments of λ_7 are given by

$$E(\lambda_7^h) = \left(\frac{e}{N_1}\right)^{phN_1} \frac{1}{(1+h)^{pN_1(1+h)}} \prod_{i=1}^{p} \frac{\Gamma(N_1 - i + N_1 h)}{\Gamma(N_1 - i)} \; . \qquad (9.2)$$

Using the first four moments, Chang, Krishnaiah and Lee (1975, 1977) approximated the distribution of $\lambda_7^{1/b}$ with Pearson Type I distribution where b is a suitably chosen integer. This approximation is used to compute the values of c_7 for $M = n_1 - p - 1 = 1(1)20(2)30$, $n_1 = N_1 - 1$ and $p = 2,3,4,5,6$ where

$$P[\tilde{\lambda}_7 \le c_7 | H_7] = (1-\alpha).$$

and $\tilde{\lambda}_7 = -2 \log \lambda_7$. These values are given in Table 13 for $\alpha = 0.05, 0.01$.

10. Applications in Time Series in the Frequency Domain

In this section, we discuss as to how the likelihood ratio test procedures on the covariance matrices of the complex multivariate normal populations can be used in the area of inference on multiple time series.

Let $\underset{\sim}{X}'(t) = (\underset{\sim}{X}_1'(t), \ldots, \underset{\sim}{X}_q'(t))$ $(t = 1,\ldots, T)$ form a Gaussian, stationary, multiple time series with zero means and covariance matrix $R(s) = (R_{jk}(s))$ where $R_{jk}(s) = E\{\underset{\sim}{X}_j(t)\ \underset{\sim}{X}_k'(t + s)\}$ and $\underset{\sim}{X}_j(t)$ is of order $p_j \times 1$. The spectral density matrix of the above time series is given by $F(\omega) = (F_{\ell j}(\omega))$ where

$$F_{\ell j}(\omega) = \frac{1}{2\pi} \sum_{s=-\infty}^{\infty} e^{-i\omega s} R_{\ell j}(s) \quad . \tag{10.1}$$

A well known estimate (e.g. see Brillinger (1974) of $F(\omega)$ is $\hat{F}(\omega) = (\hat{F}_{\ell j}(\omega))$ where

$$\hat{F}_{\ell j}(\omega) = \frac{1}{(2m+1)} \sum_{r=-m}^{m} I_{\ell j} \left(\omega + \frac{2\pi r}{T}\right) \tag{10.2}$$

and m is a suitably chosen integer. In Eq. (10.2)

$$\underset{\sim}{Z}_\ell(\lambda) = \frac{1}{\sqrt{2\pi T}} \sum_{t=1}^{T} \underset{\sim}{X}_\ell(t) \exp(-it\lambda),$$

and

$$I_{\ell j}(\lambda) = \underset{\sim}{Z}_\ell(\lambda)\ \bar{\underset{\sim}{Z}}_\ell'(\lambda).$$

Goodman (1963b) and Wahba (1968) showed that $A(\omega) = (2m+1)\ \hat{F}(\omega)$ is approximately distributed as complex Wishart distribution with (2m+1) degrees of freedom and $E(\hat{F}(\omega)) = F(\omega)$.

Now, let $H_2(\omega)$, $H_3(\omega)$ and $H_4(\omega)$ denote the following hypotheses:

$$H_2(\omega): F_{\ell j}(\omega) = 0 \quad (\ell \neq j = 1, \ldots, q)$$

$$H_3(\omega): F(\omega) = \sigma^2 F_o(\omega) \tag{10.3}$$

$$H_4(\omega): F(\omega) = F_o(\omega)$$

where σ^2 is unknown and $F_o(\omega)$ is known. Let the statistics $L_2(\omega)$, $L_3(\omega)$, and $L_4(\omega)$ be defined as follows:

$$L_2(\omega) = \frac{|A(\omega)|}{\prod_{i=1}^{q} |A_{ii}(\omega)|} \tag{10.4}$$

$$L_3(\omega) = \frac{|A(\omega)\ F_o^{-1}(\omega)|}{\{\mathrm{tr}\ A(\omega)\ F_o^{-1}(\omega)/s\}^s} \tag{10.5}$$

$$L_4(\omega) = \left(e/(2m+1)\right)^{s(2m+1)} \quad |A(\omega)\ F_o^{-1}(\omega)|^{(2m+1)} \tag{10.6}$$

$$\times\ \mathrm{etr}\ (-A(\omega)\ F_o^{-1}(\omega)).$$

where $(2m+1)A_{ij}(\omega) = \hat{F}_{ij}(\omega)$. Also, let $\tilde{L}_i(\omega) = -2 \log L_i(\omega)$ for $i = 2,3,4$.

The hypothesis $H_2(\omega)$ is accepted or rejected according as

$$\tilde{L}_2(\omega) \lesseqgtr d_2$$

where

$$P[\tilde{L}_2(\omega) \leq d_2 | H_2(\omega)] = (1 - \alpha). \tag{10.7}$$

We accept or reject $H_3(\omega)$ according as

$$\tilde{L}_3(\omega) \lesseqgtr d_3 \tag{10.8}$$

where

$$P[\tilde{L}_3(\omega) \leq d_3 | H_3(\omega)] = (1 - \alpha). \tag{10.9}$$

Similarly, the hypothesis $H_4(\omega)$ is accepted or rejected according as

$$\tilde{L}_4(\omega) \lessgtr d_4$$

where

$$P[\tilde{L}_4(\omega) \leq d_4 | H_4(\omega)] = (1 - \alpha). \tag{10.10}$$

Since $A(\omega)$ is approximately distributed as the complex Wishart matrix, approximate values of d_2, d_3, and d_4 can be obtained from Table 8, Table 9, and Table 10, respectively.

Next, let $H_2 = \bigcap_{j=1}^{k} H_2(\omega_j)$, $H_3 = \bigcap_{j=1}^{k} H_3(\omega_j)$ and $H_4 = \bigcap_{j=1}^{k} H_4(\omega_j)$

where $\omega_1,\ldots,\omega_k$ are widely separated. Then, we accept or reject H_2 according as

$$T_2 \lessgtr d_5 \tag{10.11}$$

where

$$P[T_2 \leq d_5 | H_2] = (1-\alpha) \tag{10.12}$$

and $T_2 = \max\ (\tilde{L}_2(\omega_1),\ldots,\tilde{L}_2(\omega_k))$. An alternative procedure is to accept or reject H_2 according as

$$T_2^* \lessgtr d_6 \tag{10.13}$$

where

$$P[T_2 \leq d_6 | H_2] = (1-\alpha), \tag{10.14}$$

and

$$T_2^* = \prod_{j=1}^{k} \tilde{L}_2(\omega_j).$$

Since $\omega_1,\ldots,\omega_k$ are widely separated, $L_2(\omega_1)\ldots$, $L_2(\omega_k)$ are distributed independently. So the critical values d_5 and d_6 can be computed by using the methods discussed in this chapter. We can propose similar procedures to test H_3 and H_4.

Let H_5 $F(\omega_1) = \ldots = F(\omega_k)$, where the frequencies $\omega_1, \ldots, \omega_k$ are "sufficiently" wide apart. Also, let $\tilde{L}_5(\omega) = -2 \log L_5$, where

$$L_5 = \frac{\prod_{i=1}^{k} |\hat{F}(\omega_i)|^{(2m+1)}}{|\sum_{i=1}^{k} \hat{F}(\omega_i)/k|^{k(2m+1)}}$$

Then, we accept or reject H_5 accordingly as $\tilde{L}_5 \lessgtr d_5^*$ where

$$P[\tilde{L}_5 \leq d_5^* | H_5] = (1-\alpha).$$

Since $\omega_1, \ldots, \omega_k$ are "sufficiently" wide apart, $\hat{F}(\omega_1), \ldots, \hat{F}(\omega_k)$ are distributed independently. Also, $(2m+1)$ $\hat{F}(\omega_i)$ is distributed approximately as the complex Wishart matrix with $(2m+1)$ degrees of freedom for $i=1,\ldots,k$. Hence, the values of d_5^* can be obtained from Table 11.

Next, let $\underset{\sim}{X}_i'(t) = (\underset{\sim}{X}_{i1}'(t), \ldots, \underset{\sim}{X}_{iq}'(t))$, $(t=1,\ldots,T_i)$ be a Gaussian, stationary, multiple time series with zero means and covariance matrix $R_i(s)$ and spectral density matrix $F_i(\omega)$, where $R_i(s) = (R_{iuv}(s))$ and $R_{iuv}(s) = E\{\underset{\sim}{X}_{iu}(t)\ \underset{\sim}{X}_{iv}'(t+s)\}$.

Also, let $\underset{\sim}{X}_1(t), \ldots, \underset{\sim}{X}_k(t)$ be distributed independently and $\underset{\sim}{X}_i(t)$ be of order px1 for $i=1,\ldots,k$. Let the estimate $\hat{F}_i(\omega)$ of $F(\omega)$ be defined in a similar way as $\hat{F}(\omega)$. Here, $(2m_i+1)$ $\hat{F}_i(\omega)$ is distributed approximately as the complex Wishart matrix with $2m_i+1$ degrees freedom. The hypothesis $H_6(\omega)$: $F_1(\omega) = \ldots = F_k(\omega)$ is tested as follows. We accept or reject H_6 accordingly as

$$\tilde{L}_6(\omega) \lessgtr d_6^*$$

where

$$P[\tilde{L}_6(\omega) \leq d_6^*|H_6] = (1-\alpha),$$

$$\tilde{L}_6(\omega) = -2 \log L_6(\omega),$$

$$L_6(\omega) = \frac{\prod_{i=1}^{k} |\hat{F}_i(\omega)|^{(2m_i+1)}}{\left|\sum_{i=1}^{k} (2m_i+1)\hat{F}_i(\omega)/m_o\right|^{m_o}}$$

and $m_o = 2(\sum_{i=1}^{k} m_i) + k$. The critical values d_6^* can be obtained from Table 11 when the m_i's are equal.

We will now illustrate the usefulness of some of the tables in this paper with vibration data on a C-5A transport aircraft.

Vibration measurements have been taken on the cargo deck of a C-5A transport aircraft to provide information about the dynamic environments that cargo must survive in transit and to understand better the distribution and transmission of vibrational energy throughout the aircraft structure. Measurements have been taken over certain periods by locating accelerometers at different locations on the cargo deck. We will treat each location as a variable. Data on the following variables were taken:

Variables	Longitudinal Location	Lateral Location	Directional Orientation
1 (FRV)	Forward	Right	Vertical
2 (FRL)	Forward	Right	Lateral
3 (FLV)	Forward	Left	Vertical
4 (FLL)	Forward	Left	Lateral
5 (ARV)	Aft	Right	Vertical
6 (ARL)	Aft	Right	Lateral
7 (ALV)	Aft	Left	Vertical
8 (ALL)	Aft	Left	Lateral

The basic unit of measurement is the acceleration due to gravity (1g = 980 cm/sec^2). Let the spectral density of the data on the above 8 variables at frequency ω be denoted by $\hat{F}(\omega)$ and let the corresponding population spectral density matrix be denoted by $F(\omega)$. The sample spectral density matrix at frequency ω_j is $\hat{F}(\omega_j) = S_{jo} + iS_{j1}$ where $\omega_1 = 0.15907$HZ, $\omega_a = a\omega_1$, $a = 2,3,4,5$ and

$$S_{10} = \begin{bmatrix} .3112000 & .0027030 & .3204000 & .0033170 & .4227000 & -.0007585 & .4451000 & -.0052180 \\ .0027030 & .0014000 & .0026990 & -.0004533 & .0025720 & .0027410 & .0015100 & -.0031180 \\ .3264000 & .0026990 & .3316000 & .0036660 & .4340000 & -.0008463 & .4608000 & -.0035720 \\ .0033170 & -.0004533 & .0036660 & .0004898 & .0047820 & -.0020820 & .0073780 & .0021640 \\ .4227000 & .0025720 & .4340000 & .0047820 & .6175000 & -.0051030 & .6330000 & -.0034100 \\ -.0007585 & .0027410 & -.0008463 & -.0020820 & -.0051030 & .0080290 & -.0098130 & -.0030220 \\ .4451000 & .0015100 & .4608000 & .0073780 & .6339000 & -.0098130 & .7303000 & .0008291 \\ -.0052150 & -.0031160 & -.0035720 & .0021640 & -.0034100 & -.0030220 & .0008291 & .0104800 \end{bmatrix}$$

$$S_{11} = \begin{bmatrix} 0.0000000 & -.0044000 & -.0104100 & .0004039 & .0517000 & -.0152700 & .0657400 & .0159700 \\ .0044000 & 0.0000000 & .0043750 & .0003763 & .0061800 & -.0005155 & .0102700 & .0001463 \\ .0104100 & -.0043750 & 0.0000000 & .0007700 & .1081000 & -.0172500 & .0805500 & .0181630 \\ -.0004039 & -.0003763 & -.0007700 & 0.0000000 & .0015310 & -.0007427 & -.0010273 & .0010250 \\ -.0517000 & -.0061800 & -.1081000 & -.0013310 & 0.0000000 & -.0191800 & -.0460900 & .0216900 \\ .0152700 & .0005155 & .0172500 & .0007427 & .0191800 & 0.0000000 & .0296300 & -.0011300 \\ -.0657400 & -.0102700 & -.0805500 & .0013270 & .0463900 & -.0296300 & 0.0000000 & .0331330 \\ -.0159700 & -.0001463 & -.0181600 & -.0010250 & -.0216900 & .0011300 & -.0331300 & 0.0000000 \end{bmatrix}$$

$$S_{20} = \begin{bmatrix} .0700800 & .0005063 & .0729760 & -.0003763 & .1112000 & .0004678 & .1064000 & -.0016200 \\ .0005609 & .0004357 & .0007168 & -.0003547 & -.0001567 & .0003975 & -.0002877 & -.0003964 \\ .0725700 & .0007161 & .0755300 & -.0003989 & .1123000 & .0006673 & .1096000 & -.0018970 \\ -.0003743 & -.0003547 & -.0003989 & .0004464 & -.0007892 & -.0003550 & -.0010040 & .0003924 \\ .1112000 & -.0001567 & .1123000 & -.0007892 & .2341000 & .0000914 & .2381000 & -.0020400 \\ .0004675 & .0003375 & .0006573 & -.0003550 & .0001914 & .0013360 & -.0014520 & -.0014630 \\ .1064000 & -.0002877 & .1095000 & -.0013090 & .2381000 & -.0014520 & .2754000 & -.0005648 \\ -.0016200 & -.0003964 & -.0018970 & .0003924 & -.0020400 & -.0014630 & -.0005645 & .0017120 \end{bmatrix}$$

$$S_{21} = \begin{bmatrix} 0.0000000 & .0010550 & -.0024270 & -.0024020 & .0413600 & .0023660 & .0465300 & -.0027390 \\ -.0010550 & 0.0000000 & -.0011480 & .0000026 & -.0022600 & .0001395 & -.0020950 & -.0002051 \\ .0024270 & .0011480 & 0.0000000 & -.0025040 & .0463300 & .0022950 & .0524000 & -.0026480 \\ .0024020 & -.0000026 & .0025040 & 0.0000000 & .0041830 & -.0000834 & .0038660 & .0000812 \\ -.0413600 & .0022600 & -.0463300 & -.0041830 & 0.0000000 & .0042450 & .0130200 & -.0050250 \\ -.0023660 & -.0001395 & -.0022950 & .0000834 & -.0042450 & 0.0000000 & -.0029430 & -.0000308 \\ -.0465300 & .0020950 & -.0524000 & -.0038660 & -.0130200 & .0029430 & 0.0000000 & -.0038830 \\ .0027390 & .0002051 & .0026480 & -.0000812 & .0051250 & .0000308 & .0038830 & 0.0000000 \end{bmatrix}$$

$$S_{30} = \begin{bmatrix} .0164600 & .0003811 & .0185700 & -.0005107 & .0143400 & .0011710 & .0220500 & -.0015350 \\ .0003811 & .0007633 & .0009133 & -.0006139 & .0034630 & -.0002562 & .0039860 & .0003206 \\ .0185700 & .0009133 & .0196600 & -.0006396 & .0147100 & .0012600 & .0225900 & -.0015980 \\ -.0005107 & -.0006139 & -.0006396 & .0003513 & -.0024510 & .0001858 & -.0037240 & -.0002224 \\ .0143400 & .0034630 & .0147100 & -.0024510 & .1321000 & .0014900 & .1422600 & -.0042980 \\ .0011710 & -.0002562 & .0012600 & .0001858 & .0014900 & .0015540 & .0018380 & -.0017730 \\ .0220500 & .0039860 & .0225900 & -.0037240 & .1422600 & .0018380 & .1831000 & -.0045580 \\ -.0015350 & .0003206 & -.0015980 & -.0002224 & -.0042980 & -.0017730 & -.0045580 & .0021430 \end{bmatrix}$$

$$S_{31} = \begin{bmatrix} 0.0000000 & -.0000645 & .0012290 & .0004898 & -.0022730 & .0014420 & .0030740 & -.0016530 \\ .0000645 & 0.0000000 & .0002628 & .0000029 & .0021210 & .0000690 & .0011520 & -.0007933 \\ -.0012290 & -.0002628 & 0.0000000 & .0001996 & .0021690 & .0013270 & .0077690 & -.0015220 \\ -.0004898 & -.0000029 & -.0001996 & 0.0000000 & -.0025050 & -.0006137 & -.0019060 & .0007250 \\ .0022730 & -.0021210 & -.0021090 & .0025050 & 0.0000000 & .0065620 & -.0264800 & -.0080510 \\ -.0014420 & -.0000690 & -.0013270 & .0006137 & -.0065620 & 0.0000000 & -.0075690 & .0001471 \\ -.0030740 & -.0011520 & -.0077690 & .0013060 & .0264800 & .0075690 & 0.0000000 & -.0045390 \\ .0016530 & .0007933 & .0015220 & -.0007250 & .0083510 & -.0001471 & .0045390 & 0.0000000 \end{bmatrix}$$

$$
S_{40} = \begin{bmatrix}
.0175130 & .0004323 & .0173700 & -.0004332 & -.0014830 & -.0020470 & -.0035530 & .0023870 \\
.0004329 & .0010460 & .0011190 & -.0010330 & .0035700 & .0001542 & .0027770 & -.0001921 \\
.0173700 & .0011190 & .0177500 & -.0011360 & -.0041440 & -.0016660 & -.0056940 & .0016640 \\
-.0004332 & -.0010330 & -.0011360 & .0011560 & -.0033250 & .0000601 & -.0026070 & -.0000872 \\
-.0014830 & .0035700 & -.0043440 & -.0034250 & .0891900 & .0027140 & .0974400 & -.0030700 \\
-.0020470 & .0001542 & -.0016660 & .0000601 & .0027140 & .0018160 & .0031220 & -.0020710 \\
-.0035530 & .0027770 & -.0056940 & -.0026070 & .0974400 & .0031220 & .1220000 & -.0037580 \\
.0023870 & -.0001921 & .0014640 & -.0000872 & -.0033700 & -.0020710 & -.0037580 & .0024530
\end{bmatrix}
$$

$$
S_{41} = \begin{bmatrix}
0.0000000 & .0004232 & -.0022300 & -.0003321 & .0283700 & .0001471 & .0235500 & .0000247 \\
-.0004232 & 0.0000000 & -.0004769 & .0002420 & .0017030 & .0010000 & .0017070 & -.0011890 \\
.0022300 & .0004769 & 0.0000000 & -.0002708 & .0251500 & .0000198 & .0257100 & .0001436 \\
.0003321 & -.0002420 & .0002708 & 0.0000000 & -.0023520 & -.0011320 & -.0029710 & .0013630 \\
-.0283700 & -.0017030 & -.0251500 & .0023520 & 0.0000000 & .0062310 & .0078030 & -.0085380 \\
-.0001471 & -.0010000 & -.0000198 & .0011320 & -.0062310 & 0.0000000 & -.0056680 & -.0000241 \\
-.0235500 & -.0017070 & -.0257100 & .0029710 & -.0075030 & .0056680 & 0.0000000 & -.0081600 \\
-.0000247 & .0011890 & -.0001436 & -.0013630 & .0085380 & .0000241 & .0081600 & 0.0000000
\end{bmatrix}
$$

$$
S_{50} = \begin{bmatrix}
.0177990 & -.0002297 & .0194000 & .0007346 & -.0165300 & .0016770 & -.0232900 & -.0010940 \\
-.0002297 & .0000692 & -.0003287 & -.0000763 & -.0011650 & -.0004262 & -.0007752 & .0004927 \\
.0194000 & -.0003287 & .0219900 & .0010720 & -.0203600 & .0023040 & -.0240700 & -.0021060 \\
.0007346 & -.0000763 & .0010720 & .0007642 & -.0003531 & .0006507 & -.0006490 & -.0007398 \\
-.0165300 & -.0011650 & -.0203600 & -.0003531 & .0915660 & -.0017490 & .1052000 & .0004515 \\
.0016770 & -.0004262 & .0023040 & .0006507 & -.0017490 & .0019100 & -.0011770 & -.0020380 \\
-.0232900 & -.0007752 & -.0240700 & -.0006490 & .1052000 & -.0011770 & .1295000 & .0001445 \\
-.0010940 & .0004927 & -.0021060 & -.0007398 & .0003515 & -.0020380 & .0001445 & .0022160
\end{bmatrix}
$$

$$
S_{51} = \begin{bmatrix}
0.0000000 & -.0004046 & .0002579 & .0004523 & .0143100 & .0002641 & .0135000 & -.0005494 \\
.0004046 & 0.0000000 & .0003417 & -.0000041 & .0002163 & .0004148 & .0010520 & -.0004537 \\
-.0002579 & -.0003417 & 0.0000000 & .0003401 & .0162500 & -.0000514 & .0160100 & -.0002829 \\
-.0004523 & .0000041 & -.0003401 & 0.0000000 & -.0001052 & -.0004866 & -.0009598 & .0005184 \\
-.0143100 & -.0002163 & -.0162500 & .0001052 & 0.0000000 & -.0024870 & -.0100500 & .0022760 \\
-.0002641 & -.0004148 & .0000514 & .0004866 & .0024870 & 0.0000000 & .0022610 & -.0000928 \\
-.0135000 & -.0010520 & -.0160100 & .0009598 & .0100500 & -.0022010 & 0.0000000 & .0017320 \\
.0005494 & .0004537 & .0002829 & -.0005184 & -.0022760 & .0000928 & -.0017320 & 0.0000000
\end{bmatrix}
$$

Let $\hat{F}(\omega)$ be partitioned as

$$\hat{F}(\omega) = \begin{bmatrix} \hat{F}_{11}(\omega) & \hat{F}_{12}(\omega) & \hat{F}_{13}(\omega) & \hat{F}_{14}(\omega) \\ \hat{F}_{21}(\omega) & \hat{F}_{22}(\omega) & \hat{F}_{23}(\omega) & \hat{F}_{24}(\omega) \\ \hat{F}_{31}(\omega) & \hat{F}_{32}(\omega) & \hat{F}_{33}(\omega) & \hat{F}_{34}(\omega) \\ \hat{F}_{41}(\omega) & \hat{F}_{42}(\omega) & \hat{F}_{43}(\omega) & \hat{F}_{44}(\omega) \end{bmatrix}$$

where $\hat{F}_{ii}(\omega)$ is of order 2x2 for i = 1,2,3,4. We computed $\tilde{L}_2(\omega_j)$ = -2 log $L_2(\omega_j)$ where

$$L_2(\omega_j) = \frac{|(2m+1)\hat{F}(\omega_j)|}{\prod_{i=1}^{4} |(2m+1)F_{ii}(\omega_j)|},$$

and (2m+1) = 19. The values of $\tilde{L}_2(\omega_j)$ in this case are found to be 47.760, 31,667, 33.684, 35.646 and 37.738 respectively. The value of the critical value d_2 for n=19, s=8 and q=4 from Table 8 is found to be 4.217 at 5% significance level. Since the computed values of $L_2(\omega_j)$ are greater than the value from the table, we conclude that the sets (1,2), (3,4), (5,6), (7,8) of variables are not independent for each of the five frequencies considered.

Next, we computed the value of $\tilde{L}_3(\omega_j)$ = -2 log $L_3(\omega_j)$ where

$$L_3(\omega_j) = \frac{|(2m+1)\hat{F}(\omega_j)|}{\{(2m+1)\mathrm{tr}\ \tilde{F}(\omega_j)/s\}^s}$$

s=9, (2m+1) = 19. It is found that the values of $\tilde{L}_3(\omega_j)$, in this case, are 78.321, 65.267, 63,226. 59.019 and 61.852 respectively. The critical value d_3 for n=19 and s=8 is found to be 5.142 at 5% level. So, we

reject (individually) the hypotheses that $F(\omega_j) = \sigma^2 I_p$ for $j = 1,2,3,4,5$.

For the hypothesis H_5 we will consider only the first four variables. Let spectral density matrix of the data on the first four variables at frequency ω_i be devoted by $\hat{F}(\omega_i)$ and let the corresponding population spectral density matrix be denoted by $F(\omega_i)$. The sample spectral density matrices at frequencies 0.15907 Hz, 0.47721 Hz and 0.79535 Hz are $\hat{F}(0.15907) = S_{10} + i\, S_{11}$, $\hat{F}(0.47721) = S_{20} + i\, S_{21}$ and $\hat{F}(0.79535) = S_{30} + i\, S_{31}$, respectively where

$$S_{10} = \begin{bmatrix} .3112000 & .0027030 & .3204000 & .0033170 \\ .0027030 & .0014000 & .0026990 & -.0008533 \\ 3204000 & .0026990 & .3316000 & .0036668 \\ .0033170 & -.0008533 & .0036668 & .0008898 \end{bmatrix}$$

$$S_{11} = \begin{bmatrix} 0.0000000 & -.0044000 & -.0104100 & .0004039 \\ .0044000 & 0.0000000 & .0049750 & .0003763 \\ .0104100 & -.0049750 & 0.0000000 & .0007700 \\ -.0004039 & -.0003763 & -.0007700 & 0.0000000 \end{bmatrix}$$

$$S_{20} = \begin{bmatrix} .0184600 & .0008811 & .0185700 & -.0005107 \\ .0008811 & .0007630 & .0009133 & -.0006139 \\ .0185700 & .0009133 & .0196800 & -.0006396 \\ -.0005107 & -.0006139 & -.0006396 & .0005513 \end{bmatrix}$$

$$S_{21} = \begin{bmatrix} 0.0000000 & -.0006045 & .0012290 & .0004898 \\ .0006045 & 0.0000000 & .0002828 & .0000029 \\ -.0012290 & -.0002828 & 0.0000000 & .0001996 \\ -.0004898 & -.0000029 & -.0001996 & 0.0000000 \end{bmatrix}$$

$$S_{30} = \begin{bmatrix} .0177900 & -.0002297 & .0194000 & .0007346 \\ -.0002297 & .0006892 & -.0003287 & -.0006763 \\ .0194000 & -.0003287 & .0215900 & .0010720 \\ .0007346 & -.0006763 & .0010720 & .0007642 \end{bmatrix}$$

$$S_{31} = \begin{bmatrix} 0.0000000 & -.0004046 & .0002579 & .0004523 \\ .0004046 & 0.0000000 & .0003417 & -.0000041 \\ -.0002579 & -.0003417 & 0.0000000 & .0003401 \\ -.0004523 & .0000041 & -.0003401 & 0.0000000 \end{bmatrix}$$

We computed the value of $\tilde{L}_5 = -2 \log L_5$ where

$$L_5 = \frac{\prod_{i=1}^{3} |A(\omega_i)/(2m+1)|^{(2m+1)}}{|\sum_{i=1}^{3} A(\omega_i)/3(2m+1)|^{3(2m+1)}},$$

$\omega_1 = 0.15907$ Hz, $\omega_2 = 0.47721$ Hz, $\omega_3 = 0.79535$ Hz, $(2m+1) = 19$. The value of $\tilde{L}_5$ is found to be 224.72. The critical value d_5^* for $n_o=19$, $p=4$ and $k=3$ from Table ii is found to be 50.93 at the 5% significance level. Since $\tilde{L}_5$ is greater than the value from the table, we conclude that the spectral density matrices $F(\omega_1)$, $F(\omega_2)$ and $F(\omega_3)$ are significantly different from each other.

References

Anderson, T. W. (1958). An Introduction to Multivariate Statistical Analysis. Wiley, New York.

Box, G. E. P. (1949). A general distribution theory for a class of likelihood criteria. Biometrika, 36, 317-346.

Brillinger, D. R. (1974). Time Series: Data Analysis and Theory. Holt, Rinehart and Winston, Inc.

Carmeli, M. (1974). Statistical theory of energy levels and random matrices in physics. J. Statist. Phy. 10, 259-297.

Chang, T. C., Krishnaiah, P. R. and Lee, J. C. (1975). Approximations to the distributions of the likelihood ratio statistics for testing the hypotheses on covariance matrices and mean vectors simultaneously. Aerospace Research Laboratories TR 75-0176.

Chang, T. C., Krishnaiah, P. R. and Lee, J. C. (1977). Approximations to the distributions of the likelihood ratio statistics for testing the hypotheses on covariance matrices and mean vectors simultaneously. In Applications of Statistics (P. R. Krishnaiah, ed.), pp. 97-108. North-Holland Publishing Company.

Goodman, N. R. (1963a). Statistical analysis based on a certain multivariate complex Gaussian distribution (an introduction). Ann. Math. Statist. 34, 152-76.

Goodman, N. R. (1963b). The distribution of the determinant of a complex Wishart distributed matrix. Ann. Math. Statist. 34, 178-180.

Gupta, A. K. (1971). Distribution of Wilks' likelihood-ratio criterion in the complex case. Ann. Inst. Statist. Math. 23, 77-87.

Hannan, E. J. (1970). Multiple Time Series. Wiley, New York.

James, A. T. (1964). Distributions of matrix variates and latent roots derived from normal samples. Ann. Math. Statist. 35, 475-501.

Khatri, C. G. (1965). Classical statistical analysis based on a certain multivariate complex Gaussian distribution. Ann. Math. Statist. 36, 98-114.

Krishnaiah, P. R. (1976). Some recent developments on complex multivariate distributions. J. Multivariate Anal. 6, 1-30.

Krishnaiah, P. R. and Lee, J. C. (1976). On covariance structure. Sankhyā, Series A, 38, 357-371.

Krishnaiah, P. R., Lee, J. C. and Chang, T. C. (1975). On the distributions of the likelihood ratio statistics for tests of certain covariance structures of complex multivariate normal populations. Aerospace Research Laboratories TR 75-0169.

Krishnaiah, P. R., Lee, J. C. and Chang, T. C. (1976). The distributions of the likelihood ratio statistics for tests of certain covariance structures of complex multivariate normal populations. Biometrika 63, 543-549.

Lee, J. C., Krishnaiah, P. R. and Chang, T. C. (1975). Approximations to the distributions of the determinants of real and complex multivariate Beat matrices. Aerospace Research Laboratories TR 75-1068.

Lee, J. C., Krishnaiah, P. R. and Chang, T. C. (1976). On the distribution of the likelihood ratio test statistic for compound symmetry. S. African Statist. J. 10, 49-62.

Mauchly, J. W. (1940). Significance test for sphericity of a normal n-variate distribution. Ann. Math. Statist. 11, 204-209.

Nagarsenker, B. N. and Das, M. M. (1975). Exact distribution of sphericity criterion in the complex case and its percentage points. Comm. Statist. 4, 363-375.

Wahba, G. (1968). On the distributions of some statistics useful in the analysis of jointly stationary time series. Ann. Math. Statist. 39, 1849-1862.

Wilks, S. S. (1932). Certain generalization in the analysis of variance. Biometrika 24, 471-494.

Wilks, S. S. (1935). On the independence of K sets of normally distributed statistical variables. Econometrica 3, 309-325.

Wooding, R. A. (1956). The multivariate distribution of complex normal variables. Biometrika 43, 212-215.

Appendix

Tables 7 through 13 give percentage points of various distributions discussed in this chapter. These tables are useful in implementation of the likelihood ratio test procedures. A description of these tables is given below.

Table 7: Percentage Points of the Distribution of the Determinant of the Complex Multivariate Beta Matrix

The entries in this table give the values of c_1 where

$$P[C_1 \leq c_1] = (1 - \alpha),$$

$C_1 = -\{(2n+q-p)\log U\}/\chi^2_{2pq,\alpha}$ and $U = |A_1(A_1+A_2)^{-1}|$.

Here A_1 and A_2 are distributed independently as central pxp complex Wishart matrices with n and q degrees of freedom respectively. These entries in this table are reproduced from a technical report by Lee, Krishnaiah and Chang (1975).

Table 8: Percentage Points of the Distribution of $\tilde{\lambda}_2$ for Multiple Independence

The entries in this table give the values of c_2 where

$$P[\tilde{\lambda}_2 \leq c_2] = (1 - \alpha),$$

$\tilde{\lambda}_2 = -2 \log \lambda_2$ and

$$\lambda_2 = \frac{|A|}{\prod_{i=1}^{q} |A_{ii}|}$$

Here, $A = (A_{ij})$: pq×pq is the central complex Wishart matrix with n degrees of freedom and $E(A) = n\Sigma$, and $\Sigma = (\Sigma_{ij})$. Also, A_{ij} and ${}_{ij}$ are of order p×p. In addition, $\Sigma_{ij} = 0$ for $i \neq j = 1,2,\ldots,q$, $M=n-s-3$ and $s=pq$.

Krishnaiah, Lee and Chang (1976) gave the values of c_2 for $\alpha = 0.05$, $M = 1(1)4(2)16,20,24,30$, $p=1,2,3$, and $q=3,4,5$. These entries are included in Table 8 with the kind permission of Biometrika Trustees. The remaining entries in the table are reproduced from a technical report by Krishnaiah, Lee and Chang (1975).

Table 9: Percentage Points of the Likelihood Ratio Test Statistic for Sphericity

The entries in this table are the values of c_3 where

$$P[\tilde{\lambda}_3 \leq c_3] = (1 - \alpha),$$

$$\tilde{\lambda}_3 = -2 \log\{|A\Sigma_o^{-1}|/(\mathrm{tr}A\Sigma_o^{-1}/s)^s\},$$

and $M=n-s-3$. Also, $A{:}s\times s$ is distributed as the central complex Wishart matrix with n degrees of freedom and $E(A) = n\sigma^2\Sigma_o$ where σ^2 is unknown and Σ_o is known. Upper 5% points of the distribution of $\tilde{\lambda}_3$ are given in Krishnaiah, Lee and Chang (1976) for $s=7(1)10$ and $M=1(1)5,7,10,15,20,30(5)50,60$. These entries are reproduced in Table 9 with the kind permission of Biometrika Trustees. The remaining entries in the table are reproduced from a technical report by Krishnaiah, Lee and Chang (1975).

Table 10: Percentage points of the Likelihood Ratio Statistic for Specifying the Covariance Matrix

The entries in this table give the values of c_4 where

$$P[\tilde{\lambda}_4 \leq c_4] = (1 - \alpha),$$

$$\lambda_4 = (e/n)^{sn}\ |A\Sigma_o^{-1}|^n\ \mathrm{etr}(-A\Sigma_o^{-1}) \text{ and } \tilde{\lambda}_4 = -2 \log \lambda_4.$$

In the above equation, $A{:}s\times s$ is distributed as the central Wishart matrix with n degrees of freedom and $E(A) = n\Sigma_o$ where Σ_o is known. Krishnaiah, Lee and Chang (1976) gave upper 5% points of the distribution of $\tilde{\lambda}_4$ for s=2(1)7, M=1(1)5,7,10,15,20,30 where M=n-s-1. These percentage points are reproduced in Table 10 with the kind permission of the Biometrika Turstees. The remaining entries in the table are reproduced from a technical report by Krishnaiah, Lee and Chang (1975).

Table 11. Percentage Points of the Likelihood Ratio Tests Statistic for the Homogeneity of the Covariance Matrices

The entries in this table are the values of c_5 where

$$P[\tilde{\lambda}_5 \leq c_5] = (1 - \alpha),$$

$\tilde{\lambda}_5 = -2 \log \lambda_5$ and

$$\lambda_5 = \frac{\prod_{i=1}^{q} |A_{ii}/n_i|^{n_i}}{\left|\sum_{i=1}^{q} A_{ii}/n\right|^{n}}$$

where $A_{11},\ldots,A_{qq}$ are distributed independently as central $p\times p$ complex Wishart matrices with $n_1,\ldots,n_q$ degrees of freedom respectively, $E(A_{11}/n_1)=\ldots=E(A_{kk}/n_k)$ and $n=n_1+\ldots+n_q$. Upper 5% points of the distribution of $\tilde{\lambda}_5$ are given in Krishnaiah, Lee and Chang (1976) for p = 3,4; q= 2(1)6,8; n_o=5(1)20,25,30 where $n_1=\ldots=n_q=n_o$. These points are reproduced in Table 11 with the kind permission of Biometrika Trustees. The remaining entries in this table are reproduced from a technical report by Krishnaiah, Lee and Chang (1975).

Table 12: Percentage Points of the Likelihood Ratio Test Statistic for the Homogeneity of Complex Multivariate Normal Populations

The entries in this table are the values of c_6 where

$$P[\tilde{\lambda}_6 \leq c_6 | H_6] = (1 - \alpha),$$

$\tilde{\lambda}_6 = -2 \log \lambda_6$ and λ_6 is the likelihood ratio statistic for testing the hypothesis H_6 of the homogeneity of q complex p-variate normal populations. The statistic λ_6 is defined in Section 8. In the table, q denotes the number of populations $M=N_o-p$ and N_o is the common size of various groups. Chang, Krishnaiah and Lee (1977) gave upper 5% points of the distribution of $\tilde{\lambda}_6$ for q=2(1)5 and M=1(1)20,25,30. These percentage points are reproduced in Table 12 with the kind permission of the North-Holland Publishing Company. The remaining entries in this table are reproduced from the technical report by Chang, Krishnaiah and Lee (1975).

Table 13: Percentage Points of the Likelihood Ratio Test for $\Sigma = \Sigma_o$ and $\underset{\sim}{\mu} = \underset{\sim}{\mu}_o$.

The entires in this table are the values of c_7 where

$$P[\tilde{\lambda}_7 \leq c_7 | H_7] = (1 - \alpha), \quad \tilde{\lambda}_7 = -2 \log \lambda_7$$

and $\tilde{\lambda}_7$ is the likelihood ratio statistic for testing the hypothesis H_7 where

$$H_7: \Sigma_1 = \Sigma_o, \ \underset{\sim}{\mu}_1 = \underset{\sim}{\mu}_o.$$

Here $\underset{\sim}{\mu}_1$ and Σ_1 are respectively the mean vector and covariance matrix of p-variate complex normal distribution and $\underset{\sim}{\mu}_o$ and Σ_o are known. Also, $M=N_1-p-2$ and N_1 is the sample size. Chang, Krishnaiah and Lee (1977) gave the values of c_7 for $\alpha = 0.05$, M = 1(1)20(2)30 and p = 2(1)6. These values are reproduced in Table 13 with the kind permission of North-Holland Publishing Company. The remaining entries are reproduced from the technical report by Chang, Krishnaiah and Lee (1975).

Some of the entries in Tables 2-5 are reproduced from Krishnaiah, Lee and Chang (1976) with the kind permission of the Biometrika Trustees. Whereas some of the entries in Table 6 are reproduced from Chang, Krishnaiah and Lee (1977) with the kind permission of North-Holland Publishing Company.

Table 7*

Percentage Points of the Distribution of the Determinant of the Complex Multivariate Beta Matrix

	p = 3 q = 3		p = 3 q = 4		p = 3 q = 5		p = 3 q = 6		p = 3 q = 7		p = 3 q = 8	
M α	.050	.010	.050	.010	.050	.010	.050	.010	.050	.010	.050	.010
1	1.310	1.378	1.343	1.414	1.377	1.453	1.412	1.491	1.446	1.528	1.478	1.564
2	1.132	1.153	1.154	1.177	1.177	1.202	1.201	1.228	1.225	1.253	1.249	1.278
3	1.076	1.087	1.092	1.104	1.109	1.122	1.127	1.141	1.145	1.160	1.163	1.179
4	1.050	1.057	1.062	1.070	1.075	1.083	1.089	1.098	1.103	1.113	1.117	1.128
5	1.036	1.041	1.045	1.050	1.055	1.061	1.066	1.073	1.078	1.085	1.089	1.097
6	1.027	1.030	1.034	1.038	1.043	1.047	1.051	1.056	1.061	1.066	1.071	1.077
7	1.021	1.024	1.027	1.030	1.034	1.037	1.041	1.045	1.049	1.054	1.057	1.062
8	1.017	1.019	1.022	1.024	1.028	1.030	1.034	1.037	1.041	1.044	1.048	1.052
9	1.014	1.016	1.018	1.020	1.023	1.025	1.029	1.031	1.034	1.037	1.040	1.044
10	1.011	1.013	1.015	1.017	1.019	1.021	1.024	1.026	1.029	1.032	1.035	1.037
12	1.008	1.009	1.011	1.012	1.015	1.016	1.018	1.020	1.022	1.024	1.026	1.028
14	1.007	1.007	1.009	1.010	1.011	1.012	1.014	1.015	1.017	1.019	1.021	1.022
16	1.005	1.006	1.007	1.008	1.009	1.010	1.011	1.012	1.014	1.015	1.017	1.018
18	1.004	1.005	1.006	1.006	1.007	1.008	1.009	1.010	1.011	1.012	1.014	1.015
20	1.003	1.004	1.005	1.005	1.006	1.007	1.008	1.008	1.010	1.010	1.012	1.012
30	1.002	1.002	1.002	1.002	1.003	1.003	1.004	1.004	1.005	1.005	1.006	1.006
60	1.000	1.000	1.001	1.001	1.001	1.001	1.001	1.001	1.001	1.001	1.002	1.002
120	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.001	1.001
$\chi^2_{2pq,\alpha}$	28.8693	34.8053	36.4150	42.9798	43.7730	50.8922	50.9985	58.6192	58.1240	66.2062	65.1708	73.6826

Table 7 (continued)

	p = 3 q = 9		p = 3 q = 10		p = 4 q = 4		p = 4 q = 5		p = 4 q = 6		p = 4 q = 7	
M α	.050	.010	.050	.010	.050	.010	.050	.010	.050	.010	.050	.010
1	1.509	1.598	1.538	1.630	1.354	1.425	1.373	1.444	1.394	1.466	1.416	1.490
2	1.271	1.303	1.293	1.326	1.166	1.189	1.182	1.205	1.199	1.223	1.217	1.242
3	1.180	1.198	1.197	1.216	1.102	1.114	1.114	1.127	1.128	1.141	1.142	1.156
4	1.131	1.143	1.145	1.158	1.070	1.078	1.080	1.088	1.091	1.100	1.103	1.112
5	1.101	1.109	1.113	1.122	1.052	1.057	1.060	1.065	1.069	1.075	1.078	1.085
6	1.081	1.087	1.090	1.097	1.040	1.044	1.047	1.051	1.054	1.059	1.062	1.067
7	1.066	1.071	1.074	1.080	1.032	1.035	1.037	1.041	1.044	1.047	1.051	1.054
8	1.055	1.059	1.062	1.067	1.026	1.028	1.031	1.033	1.036	1.039	1.042	1.045
9	1.047	1.050	1.053	1.057	1.021	1.024	1.026	1.028	1.030	1.033	1.035	1.038
10	1.040	1.043	1.046	1.049	1.018	1.020	1.022	1.024	1.026	1.028	1.030	1.033
12	1.031	1.033	1.035	1.038	1.013	1.015	1.016	1.018	1.020	1.021	1.023	1.025
14	1.024	1.026	1.028	1.030	1.010	1.011	1.013	1.014	1.015	1.016	1.018	1.020
16	1.020	1.021	1.023	1.025	1.008	1.009	1.010	1.011	1.012	1.013	1.015	1.016
18	1.016	1.018	1.019	1.020	1.007	1.007	1.008	1.009	1.010	1.011	1.012	1.013
20	1.014	1.015	1.016	1.017	1.006	1.006	1.007	1.008	1.009	1.009	1.010	1.011
30	1.007	1.008	1.008	1.009	1.003	1.003	1.003	1.004	1.004	1.004	1.005	1.005
60	1.002	1.002	1.002	1.002	1.001	1.001	1.001	1.001	1.001	1.001	1.001	1.001
120	1.001	1.001	1.001	1.001	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
$\chi^2_{2pq,\alpha}$	72.1532	81.0688	79.0819	88.3794	46.1943	53.4858	55.7585	63.6907	65.1708	73.6826	74.4683	83.5134

Table 7 (continued)

M	α	p = 4 q = 8		p = 4 q = 9		p = 4 q = 10		p = 5 q = 5		p = 5 q = 6		p = 5 q = 7	
		.050	.010	.050	.010	.050	.010	.050	.010	.050	.010	.050	.010
1		1.440	1.514	1.463	1.539	1.485	1.562	1.378	1.448	1.390	1.458	1.403	1.472
2		1.235	1.261	1.254	1.280	1.272	1.299	1.191	1.214	1.203	1.226	1.216	1.240
3		1.157	1.171	1.172	1.187	1.186	1.202	1.123	1.135	1.133	1.145	1.144	1.157
4		1.115	1.124	1.127	1.137	1.139	1.149	1.087	1.095	1.096	1.104	1.105	1.114
5		1.088	1.095	1.098	1.106	1.108	1.116	1.066	1.072	1.073	1.079	1.081	1.087
6		1.070	1.076	1.079	1.084	1.088	1.094	1.052	1.056	1.058	1.062	1.065	1.069
7		1.058	1.062	1.065	1.070	1.073	1.077	1.042	1.045	1.047	1.051	1.053	1.057
8		1.048	1.052	1.055	1.058	1.061	1.065	1.035	1.037	1.039	1.042	1.044	1.047
9		1.041	1.044	1.047	1.050	1.053	1.056	1.029	1.031	1.033	1.035	1.038	1.040
10		1.035	1.038	1.040	1.043	1.046	1.048	1.025	1.027	1.028	1.031	1.032	1.034
12		1.027	1.029	1.031	1.033	1.035	1.037	1.019	1.020	1.022	1.023	1.025	1.026
14		1.021	1.023	1.025	1.026	1.028	1.030	1.015	1.016	1.017	1.018	1.020	1.021
16		1.017	1.018	1.020	1.021	1.023	1.024	1.012	1.013	1.014	1.015	1.016	1.017
18		1.014	1.015	1.017	1.018	1.019	1.020	1.010	1.011	1.011	1.012	1.013	1.014
20		1.012	1.013	1.014	1.015	1.016	1.017	1.008	1.009	1.010	1.010	1.011	1.012
30		1.006	1.006	1.007	1.008	1.008	1.009	1.004	1.004	1.005	1.005	1.006	1.006
60		1.002	1.002	1.002	1.002	1.002	1.003	1.001	1.001	1.001	1.001	1.002	1.002
120		1.001	1.001	1.001	1.001	1.001	1.001	1.000	1.000	1.000	1.000	1.000	1.000
$\chi^2_{2pq,\alpha}$		83.6753	93.2168	92.8083	102.8160	101.8790	112.3290	67.5048	76.1539	79.0819	88.3794	90.5312	100.4250

Table 7 (continued)

	p = 5 q = 8		p = 5 q = 9		p = 5 q = 10		p = 6 q = 6		p = 6 q = 7		p = 6 q = 8	
M α	.050	.010	.050	.010	.050	.010	.050	.010	.050	.010	.050	.010
1	1.419	1.488	1.436	1.505	1.453	1.522	1.392	1.459	1.400	1.465	1.418	1.474
2	1.230	1.254	1.245	1.269	1.259	1.284	1.210	1.233	1.219	1.242	1.230	1.252
3	1.156	1.169	1.168	1.182	1.180	1.194	1.140	1.152	1.148	1.161	1.158	1.171
4	1.115	1.124	1.125	1.135	1.136	1.145	1.102	1.110	1.110	1.118	1.118	1.126
5	1.089	1.096	1.098	1.105	1.107	1.114	1.079	1.084	1.085	1.091	1.092	1.098
6	1.072	1.077	1.079	1.084	1.087	1.092	1.063	1.067	1.068	1.073	1.075	1.079
7	1.059	1.063	1.066	1.070	1.072	1.077	1.051	1.055	1.056	1.060	1.062	1.066
8	1.050	1.053	1.056	1.059	1.062	1.065	1.043	1.046	1.047	1.050	1.052	1.055
9	1.042	1.045	1.048	1.050	1.053	1.056	1.037	1.039	1.040	1.043	1.045	1.047
10	1.037	1.039	1.041	1.044	1.046	1.049	1.031	1.034	1.035	1.037	1.039	1.041
12	1.028	1.030	1.032	1.034	1.036	1.038	1.024	1.026	1.027	1.029	1.030	1.032
14	1.022	1.024	1.025	1.027	1.029	1.030	1.019	1.020	1.021	1.023	1.024	1.025
16	1.018	1.019	1.021	1.022	1.024	1.025	1.015	1.017	1.018	1.018	1.020	1.021
18	1.015	1.016	1.017	1.018	1.020	1.021	1.013	1.014	1.014	1.015	1.016	1.017
20	1.013	1.013	1.015	1.015	1.017	1.017	1.011	1.011	1.012	1.013	1.014	1.014
30	1.007	1.007	1.008	1.008	1.009	1.009	1.006	1.006	1.006	1.007	1.007	1.007
60	1.002	1.002	1.002	1.002	1.003	1.003	1.002	1.002	1.002	1.002	1.002	1.002
120	1.001	1.001	1.001	1.001	1.001	1.001	1.000	1.000	1.000	1.001	1.001	1.001
$\chi^2_{2pq,\alpha}$	101.8790	112.3290	113.1450	124.1160	124.3420	135.8070	92.8083	102.8160	106.3950	117.8578	119.8710	131.1410

Table 7 (continued)

	p = 6 q = 9		p = 6 q = 10		p = 7 q = 7		p = 7 q = 8	
M α	.050	.010	.050	.010	.050	.010	.050	.010
1	1.421	1.485	1.433	1.497	1.401	1.464	1.406	1.467
2	1.241	1.264	1.253	1.276	1.225	1.247	1.232	1.254
3	1.168	1.180	1.178	1.191	1.154	1.167	1.162	1.174
4	1.126	1.135	1.135	1.144	1.115	1.124	1.122	1.130
5	1.100	1.106	1.107	1.114	1.090	1.096	1.096	1.102
6	1.081	1.086	1.088	1.093	1.073	1.077	1.078	1.083
7	1.068	1.071	1.074	1.078	1.060	1.064	1.065	1.069
8	1.057	1.061	1.063	1.066	1.051	1.054	1.055	1.058
9	1.049	1.052	1.054	1.057	1.044	1.046	1.047	1.050
10	1.043	1.045	1.047	1.050	1.038	1.040	1.041	1.043
12	1.033	1.035	1.037	1.039	1.029	1.031	1.032	1.034
14	1.027	1.028	1.030	1.031	1.023	1.025	1.026	1.027
16	1.022	1.023	1.025	1.026	1.019	1.020	1.021	1.022
18	1.018	1.019	1.021	1.022	1.016	1.017	1.018	1.019
20	1.016	1.016	1.018	1.018	1.014	1.014	1.015	1.015
30	1.008	1.009	1.009	1.010	1.007	1.007	1.008	1.008
60	1.002	1.003	1.003	1.003	1.002	1.002	1.002	1.003
120	1.001	1.001	1.001	1.001	1.001	1.001	1.001	1.001
$\chi^2_{2pq,\alpha}$	133.2570	145.0990	146.5670	158.9500	122.1080	133.4760	137.7010	149.7270

TABLE 8

Percentage Points of the Distribution $\tilde{\lambda}_2$ for Multiple Independence

q=3

M \ α	p=1			p=2			p=3		
	10%	5%	1%	10%	5%	1%	10%	5%	1%
1	1.895	2.243	3.002	4.679	5.143	6.096	7.936	8.481	9.575
2	1.607	1.902	2.543	4.085	4.488	5.314	7.061	7.542	8.584
3	1.395	1.651	2.207	3.628	3.984	4.713	6.368	6.798	7.661
4	1.233	1.459	1.949	3.264	3.584	4.238	5.802	6.193	6.974
5	1.104	1.306	1.746	2.967	3.258	3.851	5.331	5.690	6.405
6	1.001	1.184	1.581	2.722	2.987	3.530	4.933	5.265	5.925
7	.914	1.081	1.445	2.513	2.759	3.259	4.592	4.899	5.512
8	.842	.996	1.330	2.335	2.562	3.027	4.295	4.582	5.155
9	.780	.923	1.233	2.180	2.393	2.826	4.035	4.305	4.841
10	.727	.859	1.148	2.045	2.244	2.651	3.805	4.059	4.565
11	.680	.805	1.075	1.926	2.113	2.495	3.600	3.840	4.318
12	.639	.756	1.010	1.819	1.997	2.358	3.416	3.644	4.098
13	.603	.713	.952	1.724	1.892	2.235	3.251	3.467	3.898
14	.570	.675	.902	1.639	1.799	2.124	3.101	3.307	3.718
15	.542	.641	.856	1.562	1.714	2.023	2.963	3.161	3.553
16	.515	.610	.814	1.491	1.636	1.931	2.838	3.027	3.403
17	.492	.582	.776	1.427	1.566	1.848	2.723	2.904	3.265
18	.470	.556	.742	1.368	1.501	1.772	2.618	2.791	3.138
19	.450	.532	.711	1.313	1.441	1.702	2.520	2.687	3.020
20	.432	.511	.682	1.263	1.386	1.637	2.429	2.590	2.911
22	.400	.473	.631	1.174	1.288	1.521	2.265	2.416	2.715
24	.372	.439	.587	1.096	1.202	1.420	2.122	2.263	2.544
26	.347	.410	.548	1.028	1.123	1.332	1.997	2.129	2.393
28	.326	.386	.515	.968	1.062	1.254	1.886	2.010	2.259
30	.307	.364	.485	.915	1.004	1.185	1.786	1.904	2.140

TABLE 8 (Continued)

q=4

M	α	p=1			p=2			p=3		
		10%	5%	1%	10%	5%	1%	10%	5%	1%
1		2.982	3.384	4.234	7.374	7.909	8.983	12.428	13.049	14.276
2		2.563	2.908	3.635	6.542	7.012	7.956	11.237	11.794	12.891
3		2.248	2.551	3.186	5.886	6.306	7.149	10.271	10.775	11.768
4		2.003	2.272	2.837	5.352	5.733	6.496	9.466	9.927	10.836
5		1.807	2.049	2.558	4.909	5.259	5.955	8.783	9.210	10.049
6		1.645	1.866	2.328	4.536	4.859	5.500	8.196	8.593	9.373
7		1.511	1.713	2.138	4.217	4.514	5.111	7.684	8.057	8.785
[illegible]		1.397	1.583	1.976	3.939	4.217	4.774	7.236	7.584	8.269
9		1.298	1.472	1.837	3.697	3.958	4.479	6.837	7.166	7.812
10		1.213	1.376	1.716	3.483	3.729	4.219	6.480	6.792	7.404
11		1.139	1.291	1.610	3.293	3.525	3.988	6.161	6.457	7.037
12		1.073	1.216	1.517	3.122	3.342	3.782	5.872	6.153	6.705
13		1.014	1.149	1.434	2.969	3.178	3.596	5.608	5.878	6.404
14		.961	1.090	1.359	2.830	3.029	3.427	5.369	5.625	6.129
15		.914	1.036	1.292	2.704	2.894	3.273	5.148	5.395	5.878
16		.871	.987	1.231	2.588	2.770	3.133	4.946	5.182	5.646
17		.831	.943	1.176	2.482	2.656	3.005	4.758	4.986	5.432
18		.796	.902	1.125	2.384	2.552	2.886	4.585	4.805	5.234
19		.763	.865	1.079	2.294	2.455	2.777	4.424	4.636	5.050
20		.733	.831	1.036	2.211	2.366	2.676	4.274	4.479	4.878
22		.680	.770	.960	2.061	2.205	2.493	4.004	4.194	4.568
24		.633	.717	.894	1.930	2.064	2.335	3.765	3.944	4.297
26		.593	.672	.838	1.814	1.941	2.195	3.553	3.723	4.054
28		.557	.631	.787	1.712	1.832	2.071	3.364	3.525	3.839
30		.525	.595	.742	1.621	1.734	1.961	3.195	3.346	3.645

TABLE 8 (Continued)

q=5

M \ α	p=1 10%	p=1 5%	p=1 1%	p=2 10%	p=2 5%	p=2 1%	p=3 10%	p=3 5%	p=3 1%
1	4.169	4.617	5.543	10.270	10.860	12.030	17.180	17.862	19.194
2	3.623	4.011	4.811	9.222	9.746	10.784	15.717	16.330	17.534
3	3.206	3.549	4.253	8.379	8.852	9.787	14.506	15.872	16.165
4	2.877	3.183	3.813	7.683	8.115	8.968	13.483	14.002	15.012
5	2.609	2.886	3.457	7.098	7.496	8.280	12.603	13.086	14.024
6	2.388	2.642	3.163	6.599	6.967	7.694	11.837	12.289	13.166
7	2.202	2.435	2.914	6.167	6.510	7.188	11.162	11.587	12.412
8	2.042	2.259	2.703	5.789	6.110	6.745	10.563	10.965	11.743
9	1.904	2.106	2.521	5.456	5.758	6.355	10.028	10.407	11.144
10	1.784	1.973	2.361	5.159	5.444	6.009	9.547	9.906	10.606
11	1.678	1.856	2.221	4.893	5.164	5.699	9.109	9.452	10.119
12	1.585	1.752	2.096	4.654	4.912	5.420	8.713	9.041	9.677
13	1.500	1.659	1.985	4.438	4.683	5.167	8.349	8.663	9.273
14	1.425	1.575	1.885	4.241	4.475	4.938	8.015	8.316	8.901
15	1.357	1.500	1.794	4.061	4.285	4.727	7.708	7.997	8.559
16	1.295	1.431	1.713	3.895	4.111	4.534	7.423	7.702	8.242
17	1.238	1.369	1.638	3.743	3.950	.357	7.159	7.428	7.943
18	1.186	1.312	1.569	3.602	3.801	.192	6.914	7.173	7.676
19	1.139	1.259	1.506	3.472	3.663	4.041	6.685	6.935	7.420
20	1.095	1.210	1.448	3.350	3.536	3.900	6.471	6.713	7.183
22	1.016	1.124	1.344	3.132	3.305	3.645	6.082	6.309	6.750
24	.948	1.048	1.254	2.940	3.102	3.421	5.738	5.952	6.367
26	.889	.983	1.176	2.771	2.924	3.224	5.430	5.633	6.026
28	.836	.925	1.106	2.620	2.764	3.048	5.155	5.347	5.720
30	.790	.873	1.045	2.485	2.622	2.891	4.906	5.089	5.444

TABLE 9

Percentage Points of the Likelihood Ratio Statistic for the Sphericity Test of Complex Covariance Matrix

$\alpha=0.05$

M	s=2	s=3	s=4	s=5	s=6	s=7	s=8	s=9	s=10
1	1.496	2.648	3.866	5.157	5.512	7.921	9.375	10.870	12.399
2	1.255	2.257	3.338	4.499	5.731	7.022	8.366	9.755	11.183
3	1.080	1.968	2.939	3.993	5.122	6.314	7.562	8.860	10.199
4	.949	1.745	2.626	3.592	4.633	5.740	6.906	8.122	9.383
5	.846	1.568	2.374	3.265	4.231	5.264	6.357	7.502	8.693
6	.763	1.423	2.166	2.993	3.895	4.863	5.891	6.973	8.108
7	.696	1.303	1.992	2.763	3.608	4.519	5.490	6.514	7.586
8	.639	1.201	1.844	2.567	3.362	4.222	5.142	6.115	7.137
9	.590	1.115	1.717	2.397	3.147	3.962	4.836	5.762	6.736
10	.549	1.040	1.606	2.248	2.959	3.732	4.565	5.448	6.381
11	.512	.975	1.509	2.116	2.791	3.528	4.322	5.168	6.061
12	.481	.916	1.423	1.999	2.642	3.346	4.104	4.915	5.773
13	.453	.866	1.346	1.895	2.508	3.181	3.908	4.686	5.511
14	.429	.820	1.277	1.801	2.387	3.031	3.730	4.478	5.272
15	.406	.779	1.215	1.715	2.277	2.896	3.567	4.287	5.053
16	.386	.741	1.158	1.638	2.177	2.772	3.419	4.113	4.851
17	.368	.707	1.107	1.567	2.086	2.659	3.281	3.952	4.666
18	.352	.676	1.060	1.503	2.001	2.554	3.155	3.803	4.494
19	.336	.648	1.016	1.443	1.924	2.457	3.038	3.665	4.335
20	.322	.622	.977	1.387	1.852	2.367	2.930	3.537	4.186
22	.298	.576	.906	1.289	1.724	2.207	2.735	3.306	3.918
24	.277	.536	.845	1.204	1.612	2.066	2.564	3.103	3.683
26	.258	.501	.791	1.129	1.513	1.943	2.413	2.925	3.474
28	.242	.471	.744	1.063	1.427	1.833	2.280	2.765	3.287
30	.228	.444	.702	1.005	1.349	1.735	2.160	2.622	3.120
35	.199	.388	.616	.883	1.188	1.531	1.909	2.322	2.769
40	.177	.345	.548	.787	1.061	1.370	1.711	2.084	2.488
45	.159	.311	.494	.710	.959	1.239	1.550	1.891	2.259
50	.144	.282	.450	.647	.875	1.132	1.417	1.730	2.069
60	.122	.239	.381	.549	.744	.964	1.209	1.478	1.771

TABLE 9 (Continued)

$\alpha = 0.01$

M	s=2	s=3	s=4	s=5	s=6	s=7	s=8	s=9	s=10
1	2.174	3.440	4.746	6.112	7.531	8.995	10.501	12.041	13.612
2	1.823	2.930	4.095	5.326	6.620	7.967	9.361	10.794	12.265
3	1.570	2.554	3.602	4.724	5.912	7.158	8.455	9.797	11.176
4	1.378	2.263	3.218	4.247	5.346	6.505	7.716	8.976	10.277
5	1.229	2.033	2.908	3.859	4.880	5.962	7.101	8.286	9.517
6	1.109	1.845	2.653	3.537	4.491	5.506	6.579	7.700	8.866
7	1.010	1.689	2.439	3.265	4.160	5.117	6.129	7.192	8.302
8	.927	1.557	2.258	3.032	3.875	4.779	5.739	6.750	7.807
9	.857	1.445	2.102	2.831	3.627	4.484	5.397	6.360	7.369
10	.797	1.348	1.966	2.655	3.409	4.223	5.093	6.013	6.979
11	.745	1.263	1.847	2.499	3.216	3.992	4.823	5.703	6.629
12	.699	1.188	1.741	2.361	3.044	3.785	4.579	5.423	6.313
13	.658	1.121	1.647	2.238	2.889	3.598	4.360	5.170	6.025
14	.622	1.062	1.563	2.126	2.750	3.429	4.160	4.940	5.764
15	.590	1.009	1.486	2.026	2.624	3.276	3.979	4.730	5.525
16	.560	.960	1.418	1.934	2.508	3.135	3.813	4.537	5.304
17	.534	.917	1.355	1.850	2.402	3.007	3.660	4.359	5.101
18	.510	.877	1.297	1.774	2.305	2.888	3.520	4.195	4.913
19	.488	.840	1.244	1.703	2.216	2.779	3.389	4.043	4.738
20	.468	.806	1.195	1.638	2.133	2.677	3.267	3.901	4.576
22	.432	.746	1.108	1.522	1.985	2.495	3.050	3.646	4.283
24	.402	.694	1.034	1.421	1.856	2.337	2.860	3.423	4.025
26	.375	.650	.968	1.333	1.743	2.197	2.691	3.225	3.797
28	.352	.610	.910	1.255	1.643	2.072	2.542	3.049	3.593
30	.331	.575	.859	1.186	1.554	1.962	2.409	2.892	3.410
35	.289	.503	.753	1.042	1.368	1.731	2.129	2.561	3.025
40	.256	.447	.671	.929	1.222	1.549	1.908	2.298	2.719
45	.230	.402	.604	.838	1.104	1.401	1.729	2.085	2.469
50	.209	.366	.550	.764	1.007	1.280	1.580	1.907	2.261
60	.177	.309	.466	.649	.857	1.090	1.348	1.630	1.936

TABLE 10

Percentage Points of the Likelihood Ratio Statistic for Specifying the Covariance Matrix

$\alpha = 0.05$

M	s=2	s=3	s=4	s=5	s=6	s=7	s=8	s=9	s=10
1	11.31	21.63	35.43	52.70	73.55	98.03	126.19	158.02	193.65
2	10.86	20.57	33.44	49.60	69.16	92.18	118.72	148.79	182.49
3	10.58	19.88	32.17	47.60	66.26	88.25	113.62	142.40	174.68
4	10.40	19.42	31.29	46.18	64.19	85.40	109.87	137.65	168.85
5	10.27	19.07	30.64	45.12	62.62	83.22	106.98	134.00	164.28
6	10.18	18.82	30.14	44.30	61.40	81.50	104.69	131.04	160.60
7	10.10	18.61	29.75	43.64	60.40	80.11	102.83	128.62	157.57
8	10.04	18.45	29.43	43.11	59.58	78.96	101.28	126.61	155.01
9	9.99	18.32	29.16	42.66	58.91	77.98	99.96	124.87	152.84
10	9.95	18.21	28.94	42.28	58.32	77.15	98.82	123.41	150.97
11	9.91	18.11	28.74	41.95	57.52	76.43	97.84	122.11	149.30
12	9.88	18.02	28.58	41.67	57.38	75.80	96.98	120.98	147.87
13	9.85	17.95	28.44	41.42	57.00	75.24	96.21	119.97	146.57
14	9.83	17.89	28.31	41.21	56.67	74.75	95.54	119.06	145.44
15	9.81	17.83	28.20	41.01	56.33	74.31	94.92	118.27	144.39
16	9.79	17.79	28.10	40.84	56.09	73.91	94.38	117.55	143.44
17	9.78	17.74	28.01	40.68	55.84	73.55	93.88	116.87	142.60
18	9.77	17.70	27.92	40.54	55.61	73.23	93.43	116.29	141.81
19	9.75	17.66	27.85	40.41	55.41	72.93	93.01	115.74	141.11
20	9.74	17.63	27.78	40.29	55.23	72.66	92.64	115.22	140.44
22	9.72	17.57	27.66	40.08	54.90	72.17	91.96	114.32	139.27
24	9.70	17.52	27.56	39.90	54.61	71.76	91.39	113.54	138.25
26	9.68	17.48	27.47	39.75	54.38	71.40	90.88	112.86	137.36
28	9.67	17.44	27.40	39.61	54.17	71.09	90.45	112.27	136.61
30	9.66	17.41	27.33	39.49	53.97	70.81	90.05	111.73	135.91

TABLE 10(Continued)

$\alpha = 0.01$

M	s=2	s=3	s=4	s=5	s=6	s=7	s=8	s=9	s=10
1	15.90	27.97	43.50	62.62	85.36	111.79	141.94	175.84	213.42
2	15.23	26.46	40.91	58.73	80.00	104.77	133.08	165.01	200.54
3	14.84	25.53	39.29	56.25	76.51	100.11	127.13	157.63	191.59
4	14.58	24.91	38.18	54.51	74.02	96.77	122.81	152.21	184.99
5	14.39	24.47	37.37	53.23	72.15	94.24	119.52	148.06	179.91
6	14.25	24.13	36.74	52.24	70.71	92.25	116.89	144.72	175.77
7	14.14	23.86	36.25	51.44	69.54	90.63	114.77	142.02	172.38
8	14.05	23.65	35.85	50.80	68.59	89.30	112.99	139.70	169.53
9	13.98	23.47	35.52	50.26	67.79	88.18	111.48	137.78	167.15
10	13.92	23.33	35.24	49.80	67.11	87.22	110.20	136.15	165.05
11	13.88	23.20	35.00	49.41	66.51	86.39	109.09	134.71	163.21
12	13.83	23.10	34.80	49.08	56.01	85.67	108.12	133.44	161.62
13	13.80	23.00	34.63	48.78	65.56	85.03	107.25	132.28	160.20
14	13.76	22.92	34.47	48.52	65.17	84.48	106.50	131.28	158.91
15	13.73	22.85	34.33	48.29	64.82	83.96	105.82	130.42	157.78
16	13.71	22.78	34.20	48.08	64.49	83.52	105.20	129.58	156.76
17	13.68	22.72	34.09	47.89	64.21	83.11	104.64	128.86	155.81
18	13.67	22.67	33.99	47.73	63.95	82.74	104.13	128.18	154.94
19	13.65	22.62	33.90	47.57	63.71	82.39	103.66	127.58	154.14
20	13.63	22.58	33.81	47.43	53.50	82.09	103.24	126.99	153.41
22	13.60	22.51	33.67	47.18	63.12	81.53	102.48	126.00	152.15
24	13.58	22.44	33.54	46.97	62.73	81.06	101.83	125.13	151.03
26	13.55	22.39	33.43	46.79	62.51	80.65	101.27	124.37	150.03
28	13.54	22.34	33.34	46.63	62.27	80.30	100.77	123.72	149.20
30	13.52	22.30	33.26	46.49	62.05	79.98	100.33	123.14	148.44

TABLE 11

Percentage Points of Likelihood Ratio Statistic for the Homogeneity of the Covariance Matrices

α=0.10 p=2

n_o	q=2	q=3	q=4	q=5	q=6	q=7	q=8
4	10.07	16.80	22.99	28.93	34.72	40.41	46.01
5	9.49	15.94	21.89	27.60	33.17	38.63	44.02
6	9.15	15.43	21.23	26.80	32.22	37.56	42.82
7	8.92	15.08	20.78	26.26	31.60	36.84	42.01
8	8.76	14.84	20.47	25.87	31.15	36.33	41.43
9	8.63	14.65	20.23	25.58	30.31	35.93	41.00
10	8.54	14.51	20.04	25.36	30.54	35.63	40.66
11	8.46	14.40	19.89	25.18	30.33	35.39	40.39
12	8.40	14.30	19.77	25.03	30.16	35.19	40.17
13	8.35	14.23	19.67	24.91	30.02	35.03	39.98
14	8.30	14.16	19.59	24.80	29.89	34.89	39.82
15	8.27	14.10	19.51	24.71	29.79	34.77	39.69
16	8.23	14.05	19.45	24.64	29.70	34.67	39.57
17	8.21	14.01	19.39	24.57	29.62	34.57	39.47
18	8.18	13.97	19.34	24.51	29.54	34.49	39.38
19	8.16	13.94	19.30	24.46	29.48	34.42	39.29
20	8.14	13.91	19.26	24.41	29.43	34.36	39.23
25	8.06	13.79	19.11	24.23	29.22	34.12	38.96
30	8.02	13.72	19.02	24.11	29.08	33.96	38.77

α=0.10 p=3

n_o	q=2	q=3	q=4	q=5	q=6	q=7	q=8
5	21.08	35.76	49.54	62.91	76.03	88.97	101.79
6	19.56	33.49	46.58	59.30	71.77	84.09	96.30
7	18.63	32.08	44.75	57.04	69.12	81.04	92.85
8	18.00	31.12	43.49	55.50	67.31	78.97	90.51
9	17.54	30.42	42.57	54.39	65.98	77.44	88.79
10	17.20	29.89	41.88	53.54	64.98	76.28	87.48
11	16.92	29.47	41.33	52.87	64.19	75.37	86.46
12	16.71	29.14	40.90	52.32	63.56	74.63	85.62
13	16.52	28.86	40.53	51.88	63.03	74.03	84.93
14	16.38	28.63	40.23	51.50	62.59	73.52	84.36
15	16.25	28.43	39.97	51.18	62.20	73.09	83.87
16	16.14	28.26	39.74	50.91	61.38	72.71	83.44
17	16.05	28.11	39.55	50.67	61.60	72.38	83.08
18	15.96	27.98	39.38	50.46	61.34	72.10	82.75
19	15.89	27.87	39.23	50.27	61.13	71.85	82.46
20	15.82	27.77	39.09	50.11	60.33	71.62	82.22
25	15.58	27.39	38.59	49.49	60.20	70.78	81.27
30	15.42	27.14	38.26	49.09	59.73	70.23	80.64

TABLE 11 (Continued)

α=0.10 p=4

n_o	q=2	q=3	q=4	q=5	q=6	q=7	q=8
6	36.34	62.39	87.08	111.15	134.35	158.32	181.59
7	33.48	58.07	81.43	104.21	126.66	148.87	170.92
8	31.68	55.33	77.82	99.78	121.41	142.82	164.09
9	30.43	53.42	75.31	96.68	117.74	138.61	159.27
10	29.52	52.01	73.45	94.39	115.04	135.46	155.73
11	28.82	50.93	72.02	92.62	112.34	133.05	153.00
12	28.27	50.07	70.89	91.23	111.28	131.13	150.80
13	27.82	49.37	69.96	90.08	109.32	129.54	149.05
14	27.45	48.80	69.19	89.14	108.78	128.24	147.57
15	27.15	48.31	68.54	88.33	107.95	127.16	146.32
16	26.88	47.89	67.99	87.65	107.04	126.23	145.26
17	26.66	47.53	67.51	87.06	106.33	125.42	144.35
18	26.45	47.22	67.09	86.54	105.71	124.69	143.54
19	26.28	46.94	66.73	86.09	105.17	124.08	142.83
20	26.13	46.70	66.41	85.68	104.70	123.53	142.20
25	25.55	45.80	65.21	84.19	102.32	121.47	139.90
30	25.19	45.22	64.43	83.24	101.79	120.16	138.41

α=0.10 p=5

n_o	q=2	q=3	q=4	q=5	q=6	q=7	q=8
7	56.07	97.04	136.08	174.28	211.34	249.31	286.40
8	51.49	90.09	126.96	163.03	198.64	233.96	269.00
9	48.53	85.58	121.00	155.68	189.31	223.87	257.52
10	46.47	82.38	116.75	150.43	183.69	216.64	249.45
11	44.93	79.98	113.59	146.52	179.02	211.24	243.26
12	43.75	78.14	111.12	143.46	175.38	207.09	238.56
13	42.79	76.66	109.14	141.02	172.47	203.72	234.69
14	42.03	75.44	107.53	139.00	170.11	200.93	231.59
15	41.39	74.43	106.18	137.33	168.11	198.61	228.94
16	40.85	73.57	105.04	135.91	166.41	196.68	226.71
17	40.39	72.83	104.06	134.69	164.36	194.99	224.83
18	39.98	72.20	103.20	133.62	163.69	193.53	223.18
19	39.64	71.64	102.46	132.70	162.58	192.22	221.69
20	39.33	71.14	101.79	131.87	161.61	191.08	220.42
25	38.20	69.35	99.39	128.89	158.03	186.95	215.70
30	37.49	68.21	97.87	126.98	155.77	184.34	212.71

TABLE 11 (Continued)

$\alpha=0.05 \qquad p=2$

n_o	q=2	q=3	q=4	q=5	q=6	q=7	q=8
4	12.30	19.51	26.08	32.34	38.42	44.36	50.21
5	11.59	18.51	24.82	30.84	36.68	42.39	48.02
6	11.16	17.91	24.07	29.94	35.64	41.21	46.69
7	10.88	17.51	23.56	29.34	34.94	40.42	45.81
8	10.68	17.23	23.20	28.90	34.44	39.86	45.18
9	10.53	17.01	22.93	28.58	34.07	39.43	44.70
10	10.42	16.85	22.72	28.33	33.77	39.10	44.34
11	10.32	16.71	22.55	28.13	33.54	38.83	44.04
12	10.25	16.60	22.41	27.96	33.35	38.61	43.79
13	10.18	16.51	22.30	27.82	33.18	38.43	43.59
14	10.13	16.43	22.20	27.71	33.05	38.28	43.42
15	10.08	16.37	22.12	27.61	32.93	38.15	43.27
16	10.04	16.31	22.05	27.52	32.83	38.03	43.14
17	10.01	16.26	21.98	27.44	32.75	37.93	43.03
18	9.98	16.22	21.92	27.38	32.67	37.84	42.94
19	9.95	16.18	21.88	27.32	32.60	37.76	42.84
20	9.93	16.14	21.83	27.26	32.54	37.69	42.77
25	9.83	16.01	21.66	27.06	32.30	37.43	42.47
30	9.77	15.92	21.55	26.93	32.15	37.26	42.27

$\alpha=0.05 \qquad p=3$

n_o	q=2	q=3	q=4	q=5	q=6	q=7	q=8
5	24.35	39.80	54.17	68.05	81.52	94.99	108.18
6	22.57	37.24	50.91	64.10	77.02	89.73	102.28
7	21.48	35.66	48.88	61.66	74.16	86.46	98.62
8	20.75	34.58	47.50	59.99	72.19	84.23	96.11
9	20.22	33.80	46.50	58.77	70.78	82.59	94.28
10	19.82	33.21	45.74	57.85	69.68	81.35	92.88
11	19.51	32.75	45.14	57.12	68.84	80.37	91.79
12	19.26	32.37	44.66	56.53	68.15	79.59	90.93
13	19.04	32.06	44.26	56.04	67.58	78.93	90.18
14	18.87	31.81	43.92	55.64	67.10	78.39	89.55
15	18.72	31.59	43.64	55.30	66.70	77.94	89.03
16	18.60	31.40	43.40	55.00	66.35	77.54	88.58
17	18.49	31.23	43.18	54.74	66.05	77.18	88.18
18	18.39	31.09	43.00	54.51	65.78	76.88	87.85
19	18.30	30.96	42.83	54.31	65.54	76.61	87.55
20	18.23	30.85	42.68	54.13	65.34	76.37	87.28
25	17.95	30.43	42.13	53.46	64.56	75.47	86.26
30	17.76	30.15	41.78	53.03	64.34	74.89	85.60

TABLE 11 (Continued)

$\alpha=0.05$ $p=4$

n_o	q=2	q=3	q=4	q=5	q=6	q=7	q=8
6	40.71	67.84	93.36	118.15	142.48	166.51	190.31
7	37.47	63.08	87.24	110.70	133.75	156.49	179.04
8	35.43	60.08	83.34	105.96	128.17	150.09	171.85
9	34.03	57.99	80.62	102.65	124.27	145.63	166.81
10	33.00	56.45	78.63	100.19	121.39	142.31	163.04
11	32.21	55.26	77.09	98.31	119.16	139.76	160.19
12	31.59	54.33	75.87	96.82	117.40	137.74	157.83
13	31.09	53.57	74.88	95.60	115.37	136.11	156.04
14	30.68	52.94	74.05	94.59	114.77	134.73	154.51
15	30.33	52.42	73.36	93.74	113.77	133.56	153.18
16	30.03	51.96	72.77	93.02	112.92	132.58	152.06
17	29.78	51.57	72.25	92.38	112.18	131.73	151.10
18	29.56	51.23	71.81	91.84	111.53	130.98	150.25
19	29.36	50.93	71.41	91.35	110.95	130.32	149.52
20	29.18	50.67	71.06	90.92	110.45	129.73	148.86
25	28.54	49.69	69.77	89.34	108.57	127.58	146.43
30	28.14	49.05	68.94	88.32	107.37	126.20	144.85

$\alpha=0.05$ $p=5$

n_o	q=2	q=3	q=4	q=5	q=6	q=7	q=8
7	61.61	103.97	144.09	183.19	221.69	259.84	297.64
8	56.50	96.44	134.32	171.26	207.57	243.68	279.40
9	53.23	91.55	127.95	163.48	198.48	233.10	267.46
10	50.94	88.10	123.45	157.95	191.33	225.56	258.89
11	49.25	85.53	120.08	153.81	187.05	219.92	252.55
12	47.94	83.54	117.47	150.58	183.21	215.52	247.58
13	46.90	81.96	115.37	148.00	180.19	212.01	243.60
14	46.05	80.65	113.65	145.90	177.58	209.14	240.32
15	45.34	79.57	112.22	144.12	175.50	206.71	237.59
16	44.75	78.64	111.01	142.65	173.32	204.64	235.31
17	44.24	77.86	109.98	141.36	172.28	202.89	233.23
18	43.80	77.17	109.07	140.24	170.37	201.38	231.56
19	43.42	76.57	108.27	139.27	169.32	200.07	230.08
20	43.07	76.04	107.56	138.40	168.40	198.86	228.72
25	41.84	74.12	105.02	135.25	165.04	194.56	223.82
30	41.06	72.90	103.41	133.25	162.36	191.82	220.72

TABLE 12

Percentage Points of the Likelihood Ratio Test Statistic for the Homogeneity of Complex Multivariate Normal Populations

$\alpha = 0.05$ $p = 1$

M	q = 2	q = 3	q = 4	q = 5
1	8.12	12.47	16.36	20.06
2	8.03	12.49	16.50	20.30
3	7.97	12.51	16.59	20.45
4	7.94	12.52	16.64	20.55
5	7.92	12.53	16.68	20.62
6	7.91	12.53	16.71	20.67
7	7.89	12.54	16.73	20.71
8	7.89	12.55	16.75	20.74
9	7.88	12.55	16.77	20.77
10	7.87	12.56	16.78	20.79
11	7.87	12.56	16.79	20.81
12	7.86	12.56	16.80	20.82
13	7.86	12.56	16.81	20.84
14	7.86	12.56	16.81	20.85
15	7.85	12.57	16.82	20.86
16	7.85	12.56	16.83	20.87
17	7.85	12.57	16.83	20.88
18	7.85	12.57	16.84	20.89
19	7.85	12.57	16.84	20.89
20	7.85	12.57	16.85	20.90
25	7.84	12.57	16.86	20.92
30	7.84	12.58	16.87	20.94

TABLE 12 (Continued)

α = 0.05 p = 2

M	q = 2	q = 3	q = 4	q = 5
1	19.87	31.06	42.59	53.10
2	18.41	29.93	40.42	50.62
3	17.68	28.92	39.37	49.43
4	17.25	28.39	38.75	48.74
5	16.96	28.03	38.35	48.29
6	16.75	27.78	38.06	47.97
7	16.60	27.60	37.85	47.74
8	16.47	27.45	37.69	47.56
9	16.38	27.33	37.55	47.41
10	16.30	27.24	37 44	47.30
11	16.23	27.16	37.36	47.21
12	16.18	27.09	37.28	47.12
13	16.13	27.04	37.22	47.05
14	16.09	26.99	37.16	47.00
15	16.05	26.94	37.12	46.94
16	16.02	26.90	37.07	46.90
17	15.99	26.87	37.04	46.86
18	15.97	26.84	37.01	46.82
19	15.94	26.81	36.97	46.79
20	15.92	26.79	36.95	46.76
25	15.84	26.69	36.85	46.65
30	15.79	26.63	36.78	46.57

TABLE 12 (Continued)

$\alpha = 0.05$ $p = 3$

M	q = 2	q = 3	q = 4	q = 5
1	30.46	59.39	80.89	101.75
2	32.90	54.56	74.91	94.65
3	31.04	52.04	71.30	90.97
4	29.90	50.49	69.88	88.70
5	29.13	49.42	68.57	87.16
6	28.56	48.65	67.62	86.05
7	28.13	48.06	66.90	85.21
8	27.79	47.61	66.34	84.55
9	27.52	47.24	65.89	84.02
10	27.30	46.94	65.52	83.57
11	27.11	46.66	65.19	83.21
12	26.96	46.40	64.94	82.90
13	26.82	46.27	64.71	82.63
14	26.70	46.11	64.50	82.40
15	26.60	45.97	64.33	82.20
16	26.51	45.84	64.17	82.01
17	26.42	45.73	64.04	81.85
18	26.35	45.63	63.92	81.71
19	26.28	45.54	63.80	81.58
20	26.22	45.45	63.70	81.46
25	25.99	45.14	63.31	81.01
30	25.83	44.92	63.05	80.70

TABLE 12(Continued)

$\alpha = 0.05$ $\quad\quad$ $p = 4$

	$q = 2$	$q = 3$	$q = 4$	$q = 5$
1	58.06	95.91	131.68	165.49
2	51.79	87.16	120.63	153.23
3	48.40	82.37	114.58	145.98
4	46.20	79.55	110.73	141.37
5	44.77	77.23	108.05	138.13
6	43.58	75.67	106.08	135.78
7	42.83	74.40	104.57	133.95
8	42.17	73.51	103.36	132.51
9	41.63	72.74	102.39	131.33
10	41.19	72.10	101.57	130.37
11	40.81	71.56	100.89	129.54
12	40.50	71.09	100.29	128.84
13	40.22	70.70	99.80	128.22
14	39.97	70.35	99.36	127.70
15	39.76	70.03	98.97	127.23
16	39.57	69.77	98.62	126.83
17	39.41	69.52	98.31	126.45
18	39.25	69.30	98.04	126.13
19	39.12	69.10	97.79	125.82
20	38.99	68.92	97.55	125.54
25	38.51	68.22	96.67	124.50
30	38.18	67.74	96.07	123.76

TABLE 12 (Continued)

$\alpha = 0.01$ $p = 1$

M	q = 2	q = 3	q = 4	q = 5
1	11.78	16.66	20.99	25.06
2	11.65	16.69	21.15	25.34
3	11.57	16.71	21.25	25.52
4	11.53	16.72	21.32	25.63
5	11.50	16.73	21.37	25.72
6	11.48	16.74	21.40	25.78
7	11.46	16.75	21.43	25.83
8	11.45	16.75	21.46	25.87
9	11.44	16.76	21.47	25.90
10	11.43	16.76	21.49	25.92
11	11.42	16.77	21.51	25.95
12	11.42	16.77	21.52	25.96
13	11.41	16.77	21.52	25.98
14	11.41	16.77	21.53	26.00
15	11.40	16.78	21.54	26.01
16	11.40	16.78	21.55	26.02
17	11.40	16.78	21.55	26.03
18	11.40	16.78	21.56	26.04
19	11.39	16.78	21.57	26.05
20	11.39	16.78	21.57	26.06
25	11.38	16.79	21.59	26.09
30	11.38	16.79	21.60	26.11

TABLE 12 (Continued)

$\alpha = 0.01$ $p = 2$

M	$q = 2$	$q = 3$	$q = 4$	$q = 5$
1	25.93	36.78	50.55	61.80
2	23.94	36.42	47.84	58.76
3	22.96	35.27	46.55	57.32
4	22.36	34.59	45.79	56.49
5	21.99	34.15	45.30	55.96
6	21.72	33.83	44.95	55.58
7	21.51	33.60	44.70	55.30
8	21.35	33.42	44.49	55.08
9	21.23	33.27	44.34	54.92
10	21.12	33.16	44.21	54.78
11	21.04	33.05	44.10	54.66
12	20.97	32.98	44.02	54.57
13	20.90	32.90	43.94	54.49
14	20.84	32.84	43.87	54.42
15	20.80	32.79	43.82	54.36
16	20.76	32.74	43.77	54.31
17	20.72	32.70	43.72	54.26
18	20.69	32.67	43.68	54.22
19	20.66	32.63	43.64	54.18
20	20.63	32.60	43.62	54.14
25	20.53	32.48	43.49	54.02
30	20.45	32.40	43.41	53.93

TABLE 12 (Continued)

$\alpha = 0.01$ $p = 3$

M	$q = 2$	$q = 3$	$q = 4$	$q = 5$
1	45.07	69.52	92.43	114.42
2	40.46	63.71	85.31	106.12
3	38.10	60.56	81.65	101.88
4	36.66	58.80	79.40	99.26
5	35.68	57.53	77.88	97.51
6	34.98	56.62	76.79	96.23
7	34.45	55.92	75.95	95.27
8	34.03	55.38	75.31	94.52
9	33.70	54.95	74.78	93.93
10	33.41	54.59	74.36	93.44
11	33.19	54.29	73.99	93.01
12	32.99	54.04	73.69	92.67
13	32.82	53.82	73.43	92.37
14	32.67	53.62	73.20	92.10
15	32.55	53.46	73.00	91.87
16	32.43	53.31	72.83	91.67
17	32.33	53.18	72.68	91.48
18	32.24	53.05	72.53	91.32
19	32.16	52.95	72.40	91.18
20	32.09	52.86	72.29	91.05
25	31.80	52.48	71.85	90.55
30	31.60	52.23	71.54	90.19

TABLE 12 (Continued)

$\alpha = 0.01$ $\qquad$ $p = 4$

M	$q^* = 2$	$q^* = 3$	$q^* = 4$	$q^* = 5$
1	69.34	109.42	146.98	183.36
2	61.52	99.01	134.17	168.24
3	57.35	93.43	127.28	160.05
4	54.75	89.39	122.89	154.88
5	52.95	87.44	119.87	151.30
6	51.53	85.65	117.65	148.66
7	50.62	84.27	115.94	146.65
8	49.82	83.18	114.59	145.05
9	49.13	82.29	113.48	143.74
10	48.64	81.56	112.57	142.68
11	48.21	80.94	111.80	141.76
12	47.82	80.41	111.16	140.99
13	47.49	79.95	110.58	140.31
14	47.20	79.56	110.09	139.73
15	46.95	79.20	109.66	139.21
16	46.72	78.90	109.28	138.77
17	46.53	78.62	108.94	138.36
18	46.34	78.36	108.62	138.00
19	46.18	78.15	108.35	137.66
20	46.03	77.94	108.09	137.37
25	45.46	77.14	107.10	136.20
30	45.07	76.60	106.42	135.40

TABLE 13

Percentage Points of the Likelihood Ratio Test Statistic for $\Sigma=\Sigma_0$ and $\underset{\sim}{\mu}=\underset{\sim}{\mu}_0$

	α = 0.05					α = 0.01				
p	2	3	4	5	6	2	3	4	5	6
M										
1	20.33	34.13	51.51	72.40	96.91	26.52	42.19	61.37	84.17	110.61
2	19.20	32.18	48.40	68.00	91.05	24.99	39.58	57.48	78.79	103.56
3	18.51	30.91	46.33	65.10	87.12	24.05	37.94	54.98	75.27	98.92
4	18.03	30.02	44.97	63.02	84.25	23.41	36.82	53.24	72.79	95.57
5	17.63	29.37	43.91	61.45	82.09	22.96	36.00	51.95	70.93	93.03
6	17.43	28.87	43.09	60.22	80.36	22.61	35.37	50.95	69.47	91.02
7	17.22	28.47	42.43	59.23	78.96	22.34	34.87	50.15	68.30	89.41
8	17.05	28.15	41.89	58.41	77.80	22.12	34.47	49.50	67.35	88.07
9	16.32	27.89	41.44	57.73	76.84	21.94	34.14	48.96	66.54	86.96
10	16.81	27.65	41.05	57.15	76.00	21.79	33.85	48.50	65.85	85.99
11	16.71	27.46	40.73	56.64	75.29	21.65	33.62	48.11	65.27	85.17
12	16.63	27.23	40.45	56.20	74.64	21.55	33.41	47.77	64.76	84.44
13	16.55	27.15	40.20	55.82	74.09	21.46	33.23	47.48	64.31	83.81
14	16.49	27.02	39.33	55.48	73.61	21.37	33.07	47.22	63.92	83.26
15	16.44	26.91	39.79	55.17	73.15	21.30	32.93	46.98	63.56	82.74
16	16.38	25.91	39.61	54.90	72.76	21.23	32.81	46.77	63.25	82.29
17	16.34	26.72	39.45	54.65	72.40	21.17	32.69	46.58	62.96	81.89
18	16.30	26.63	39.31	54.43	72.08	21.12	32.59	46.41	62.70	81.51
19	16.26	25.56	39.19	54.22	71.78	21.07	32.50	46.26	62.46	81.16
20	16.23	26.49	39.06	54.04	71.51	21.03	32.41	46.11	62.24	80.86
22	16.17	26.37	38.85	53.71	71.02	20.95	32.26	45.87	61.86	80.31
24	16.12	26.27	38.67	53.43	70.60	20.88	32.14	45.65	61.53	79.83
26	16.07	26.18	38.52	53.18	70.25	20.83	32.03	45.47	61.25	79.42
28	16.04	26.10	38.39	52.93	69.93	20.78	31.94	45.31	61.08	79.07
30	16.01	26.03	38.27	52.79	69.65	20.74	31.85	45.17	60.79	78.74

TABLE 13 (Continued)

	α = 0.025					α = 0.10				
p	2	3	4	5	6	2	3	4	5	6
M										
1	23.05	37.74	55.91	77.68	103.07	17.46	30.39	48.75	66.65	90.15
2	21.75	35.47	52.47	72.35	96.69	16.51	28.65	44.00	62.71	84.84
3	20.95	34.04	50.24	69.68	92.44	15.92	27.53	42.21	60.09	81.25
4	20.41	33.06	48.68	67.42	89.37	15.52	26.76	40.93	58.19	78.62
5	20.01	32.32	47.51	65.72	87.03	15.23	26.18	39.98	56.76	76.61
6	19.72	31.77	46.62	64.39	85.19	15.01	25.74	39.24	55.64	75.02
7	19.49	31.33	45.89	63.32	83.69	14.83	25.39	38.65	54.73	73.73
8	19.30	30.97	45.30	62.44	82.46	14.69	25.11	38.16	53.98	72.66
9	19.14	30.67	44.82	61.70	81.42	14.58	24.87	37.76	53.36	71.76
10	19.01	30.42	44.40	61.08	80.52	14.48	24.67	37.41	52.82	71.00
11	18.90	30.21	44.05	60.54	79.76	14.39	24.50	37.12	52.37	70.33
12	18.80	30.02	43.74	60.06	79.09	14.32	24.35	36.86	51.96	69.74
13	18.72	29.86	43.46	59.65	78.49	14.26	24.22	36.64	51.61	69.22
14	18.65	29.72	43.23	59.29	77.96	14.21	24.11	36.44	51.29	68.76
15	18.58	29.60	43.02	58.96	77.50	14.16	24.01	36.26	51.02	68.35
16	18.53	29.48	42.83	58.66	77.08	14.12	23.92	36.10	50.76	67.98
17	18.48	29.38	42.65	58.40	76.69	14.08	23.84	35.96	50.53	67.64
18	18.43	29.29	42.50	58.16	76.35	14.04	23.76	35.83	50.33	67.34
19	18.39	29.21	42.36	57.94	76.02	14.01	23.70	35.71	50.14	67.07
20	18.35	29.13	42.23	57.74	75.75	13.98	23.64	35.60	49.97	66.81
22	18.28	29.00	42.01	57.39	75.22	13.93	23.53	35.41	49.67	66.36
24	18.23	28.89	41.81	57.09	74.78	13.89	23.44	35.25	49.41	65.98
26	18.18	28.79	41.64	56.82	74.39	13.85	23.36	35.11	49.18	65.65
28	18.14	28.71	41.49	56.60	74.06	13.82	23.29	34.99	48.99	65.35
30	18.10	28.63	41.36	56.39	73.77	13.79	23.23	34.88	48.81	65.09

END

DATE
FILMED

9 — 83

DTIC